Q1

The constant function $v = 1$ is in $V$, since it is a polynomial and hence can be represented exactly by a degree $k$ polynomial. We have

$$\int_{\Omega} |\nabla v|^2 \, dx = 0,$$

which breaks coercivity.
\[ \int_0^1 u' \, dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} u' \, dx, \]

\[ = \sum_{k=1}^{n} (u(x_k) - u(x_{k-1})) \]

\[ = u(1) - u(0). \]
First we note that the fundamental theorem of calculus holds for $C^0$ finite element functions, by subdividing the integral into cells or partial cells as usual. Second we note that $v'v = (v^2)'/2$. So,

\[
\int_0^1 v'v \, dx = \int_0^1 \frac{1}{2}(v^2)' \, dx = \left[ \frac{1}{2}v^2 \right]_0^1 = 0,
\]

by the boundary conditions. Then,

\[
a(v, v) = \int_0^1 ((v')^2 + v'v + v^2) \, dx = \int_0^1 ((v')^2 + v^2) \, dx.
\]

Thus we have coercivity of $a$ since $a(v, v) = \|v\|^2_{H^1}$ which is a special case of the coercivity condition $a(v, v) \leq C\|v\|^2_{H^1}$ with $C = 1$. 
For continuity,

$$|a(u, v)| \leq |(u, v)_{H^1}| + \left| \int_0^1 u' v \, dx \right|$$

$$\leq \|u\|_{H^1} \|v\|_{H^1} + \|u'\|_{L^2} \|v\|_{L^2},$$

$$\leq 2 \|u\|_{H^1} \|v\|_{H^1}.$$

For coercivity,

$$a(v, v) = \int_0^1 (v')^2 + v' v + v^2 \, dx,$$

$$= \frac{1}{2} \int_0^1 (v' + v)^2 \, dx + \frac{1}{2} \int_0^1 (v')^2 + v^2 \, dx \geq \frac{1}{2} \|v\|_{H^1}.$$
Q5

Let $V$ be a $C^0$ finite element space defined on $\Omega$. Multiply by test function $v \in V$ and integrate by parts, to obtain

$$
\int_{\Omega} \nabla v \cdot (\sigma \nabla u) \, dx = \int_{\Omega} vf \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} \sigma v \, dx = 0,
$$

Hence, the finite element discretisation requires to find $u_h \in V$, such that

$$
a(u_h, v) = (f, v), \quad \forall v \in V,
$$

where

$$
a(u, v) = \int_{\Omega} \nabla v \cdot (\sigma \nabla u) \, dx.
$$
Q5 [ctd]

For continuity,

\[ |a(u, v)| = \left| \sum_i \int_\Omega \frac{\partial u}{\partial x_i} \sigma \frac{\partial v}{\partial x_i} \, dx \right| \leq b \left| \sum_i \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \right| \]

\[ \leq b \sum_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2} \leq b \sum_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2} \sum_j \left\| \frac{\partial v}{\partial x_j} \right\|_{L^2} \]

\[ \leq b \left( \| u \|_{L^2}^2 + \sum_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 \right)^{1/2} \left( \| v \|_{L^2}^2 + \sum_j \left\| \frac{\partial v}{\partial x_j} \right\|_{L^2}^2 \right)^{1/2} , \]

\[ = b \| u \|_{H^1} \| v \|_{H^1} . \]
Q5 [cdt]

For coercivity,

$$\|\nu\|_{H^1(\Omega)}^2 \leq (1 + C^2_\Omega) |\nu|_{H^1(\Omega)}^2,$$

(From Lectures)

$$= (1 + C^2_\Omega) \int_\Omega \nabla \nu \cdot \nabla \nu \, dx,$$

$$= (1 + C^2_\Omega) \int_\Omega \frac{1}{\sigma} \sigma \nabla \nu \cdot \nabla \nu \, dx,$$

$$\leq (1 + C^2_\Omega) \frac{1}{a} \int_\Omega \sigma \nabla \nu \cdot \nabla \nu \, dx = \frac{1}{a} (1 + C^2_\Omega) a(\nu, \nu).$$

Hence, $a(\cdot, \cdot)$ is a symmetric bilinear form that is continuous and coercive on $H^1(\Omega)$, and hence a unique solution exists.
Q7

Choosing a $C^0$ finite element space $V$, we define $\tilde{V}$ as the subspace of functions vanishing on $\partial \Omega$ as above. We write $u = u^H + u^g$, where $u^H \in \tilde{V}$, and $u^g = g$ on $\partial \Omega$. Then,

$$a(u^H, v) = (f, v) - a(g, v), \quad \forall v \in \tilde{V},$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$
We have already checked coercivity and continuity of $a$ in lectures, so we just need to check continuity of $L(v)$ given by

$$L(v) = (f, v) - a(g, v).$$

We have

$$|L(v)| \leq \|f\|_{L^2} \|v\|_{L^2} + \|g\|_{H^1} \|v\|_{H^1} = (\|f\|_{L^2} + \|g\|_{H^1}) \|v\|_{H^1},$$

so it is continuous as required.