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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2025

MATH70022 Finite Elements: Numerical Analysis and Implementation

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1a (6), 1b (6), 3a(8), 3b(6), 3c(6)

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
2a (8), 2b (8), 4a (10)

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
4b (10)

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1c (8), 2c (4)

Signatures are required for the final version:

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

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This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite Elements: Numerical Analysis and Implementation

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Which of the following are unisolvent finite elements? In each case, justify your answer.

(a) K is the interval $[a, b]$. $\mathcal{P}$ are the degree 3 polynomials on K. The nodal variables are

$$N_1(v) = v(a),\ N_2(v) = v(b),\ N_3(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(a),\ N_4(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(b),$$

for all polynomials $v \in \mathcal{P}$. (6 marks)

(b) K is a triangle. $\mathcal{P}$ are the degree 5 polynomials on K. The nodal variables are

$$N_1(v) = v(z_1),\ N_2(v) = v(z_2),\ N_3(v) = v(z_3),\ N_4(v) = \frac{\partial v}{\partial x_1}(z^*),\ N_5(v) = \frac{\partial v}{\partial x_2}(z^*),$$

where x_1, x_2 are the coordinates in the triangle, z_1, z_2, z_3 are the vertices and $z^* = (z_1 + z_2 + z_3)/3$. (6 marks)

(c) K is the unit square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$. $\mathcal{P} = \text{span}(1, x, y, x^2, y^2, xy, x^2y, y^2x, y^2x^2)$. The nodal variables are point evaluations at the points $(i/2, j/2)$ for integers $0 \leq i, j \leq 2$. (8 marks)

(Total: 20 marks)

2. Let K be a triangle. For $f \in C^{k,\infty}(K)$, the averaged Taylor polynomial of degree k is defined as

$$(Q_{k,B}f)(x) = \frac{1}{|B|}\int_B \sum_{|\alpha|\leq k} D^\alpha f(y)\frac{(x-y)^\alpha}{\alpha!}\,\mathrm{d}\,y,$$

where: $B \subset K$ is a disk in $\mathbb{R}^2$ of area $|B|$, the sum is over multi-indices α of degree $|\alpha|$, $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2}}$, $\alpha! = \alpha_1!\alpha_2!$, and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}$.

(a) Let β be a multi-index with $|\beta| \leq k$. Show that

$$D^\beta(Q_{k,B}f)(x) = (Q_{k-|\beta|,B}D^\beta f)(x).$$

(8 marks)

(b) For a triangle K, we assume that for $f \in C^{k+1,\infty}$, there exists C (depending on K and k), such that

$$\|f - Q_{k,B}f\|_{L^2(K)} \leq C|f|_{H^{k+1}(K)}.$$

Show that for a multi-index β with $|\beta| \leq k+1$,

$$\|D^\beta(f - Q_{k,B}f)\|_{L^2(K)} \leq C|f|_{H^{k+1}(K)}.$$

(8 marks)

(c) Briefly, given the use of rescaling arguments to obtain error estimates that depend on the triangle sizes for the global interpolation operator, explain why it is critical that the semi-norm $|f|_{H^{k+1}(K)}$ appears in these formulae instead of the norm $\|f\|_{H^{k+1}(K)}$. (4 marks)

(Total: 20 marks)

3. In this question we consider the following boundary value problem solved on the interval $[0, 1]$,

$$\frac{\mathrm{d}^4 u}{\mathrm{d}\,x^4} - \frac{\mathrm{d}^2 u}{\mathrm{d}\,x^2} + u = f,\ u(0) = u(1) = \frac{\mathrm{d}^2 u}{\mathrm{d}\,x^2}(0) = \frac{\mathrm{d}^2 u}{\mathrm{d}\,x^2}(1) = 0. \tag{1}$$

(a) Formulate (1) as a linear variational problem, seeking $u \in V$ where

$$b(u_h, v) = F(v), \quad \forall v \in V, \tag{2}$$

where $b(\cdot, \cdot)$ is a symmetric bilinear form, and V is an appropriate subspace of $H^2([0, 1])$. (8 marks)

(b) Propose an appropriate finite element space $V_h \subset V$ to obtain a Galerkin approximation of (2). (6 marks)

(c) Assume that $\|f\|_{L^2([0,1])} < \infty$. Show that the Galerkin approximation has a unique solution u_h, and that there exists $C > 0$ such that

$$\|u_h\|_{H^2([0,1])} \leq C\|f\|_{L^2(\Omega)}.$$

You may make use of any of the stated results in the course. (6 marks)

(Total: 20 marks)

4. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Given a known C^∞ function $\alpha : \Omega \to \mathbb{R}$ with $0 < a < \alpha < b < \infty$ for given constants $a, b \in \mathbb{R}$, we consider the PDE

$$u - \nabla \cdot (\alpha \nabla u) = f, \tag{3}$$

with boundary conditions $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$ of Ω, and known $f \in L^2(\Omega)$. The variational form of this PDE seeks $u \in H^1(\Omega)$ such that

$$\int_\Omega uv + \alpha \nabla u \cdot \nabla v \, \mathrm{d}\, x = \int_\Omega fv \, \mathrm{d}\, x, \quad \forall u \in H^1(\Omega).$$

(a) Let u_h be the finite element approximation of the solution of (3) obtained using a quadratic Lagrange finite element space of degree 2. Assuming that the solution u of (3) is in $H^2(\Omega)$, show that

$$\|u_h - u\|_{H^1(\Omega)} \leq Ch^2 \|u\|_{H^3(\Omega)},$$

for some constant $C > 0$, independent of the mesh parameter h. You may assume any results stated in lectures without proof. (10 marks)

(b) Let w solve the equation

$$w - \nabla \cdot (\alpha \nabla w) = g, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega.$$

We assume that there exists $C_2 > 0$ such that for any $g \in L^2(\Omega)$,

$$|w|_{H^2(\Omega)} \leq C_2 \|g\|_{L^2(\Omega)}.$$

Under this assumption, show that

$$\|u_h - u\|_{L^2(\Omega)} \leq C_3 h^3 \|u\|_{H^3(\Omega)},$$

for some constant $C_3 > 0$. (You may assume the Galerkin orthogonality result.)

(10 marks)

(Total: 20 marks)

5. We consider the following abstract mixed linear variational problem. Find $u \in V$ and $p \in Q$ such that

$$a(u, v) + b(v, p) = F[v],\ b(u, q) = G[q], \tag{4}$$

for continuous bilinear forms a on $V \times V$ and b on $V \times Q$ respectively, and continuous linear forms $F \in V'$ and $G \in Q'$ respectively.

(a) Assume that there exists $\beta > 0$ such that

$$\inf_{0 \neq q \in Q} \sup_{0 \neq v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta.$$

Consider the operator $B : V \to Q'$ given by

$$Bv[p] = b(v, p), \quad \forall v \in V,\, p \in Q.$$

Show that we can always find $u_g \in V$ such that $Bu_g = G$. (You may assume that B^* injective is equivalent to B surjective, according to the Closed Range Theorem.) (6 marks)

(b) Define $Z \subset V$ as the kernel of B. Assume that a is coercive on V. Show that there exists $u_Z \in Z$ such that

$$a(u_Z, v) + b(v, p) = F[v] - a(u_g, v),\ \forall v \in Z.$$

(6 marks)

(c) Using the previous parts, show that there exists (u, p) solving (4).

(You may assume that $\mathrm{Im}(B^*) = (\mathrm{Ker}(B))^0$, where ^{0} indicates the polar space, according to the Closed Range Theorem.) (8 marks)

(Total: 20 marks)