BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

M70022

Finite Elements (Solutions)

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1. (a) This triple is unisolvent, because if $N_i(v) = 0$, $i = 1, 2, 3, 4$, then v is a degree 3 polynomial with two double roots, and hence is zero from Fundamental Theorem of Algebra.

seen ⇓

6, A

(b) This triple is not unisolvent, because the space of degree 5 polynomials is 6 dimensional and there are only 5 nodal variables.

sim. seen ⇓

6, A

(c) Let v be a polynomial in $\mathcal{P}$, which vanishes at all the specified points. In particular v vanishes at 3 points on the line $x = 0$. Since v restricted to that line is a quadratic polynomial in y only, v vanishes on that line. Hence, $v(x, y) = xq_3(x, y)$ where q is a cubic polynomial. Similarly, $v(x, y)$ vanishes on $y = 0$, and hence by continuity $q_3(x, y)$ vanishes there, so $q_3(x, y) = yq_2(x, y)$ where q_2 is a quadratic polynomial. Iterating this argument two more times, considering the lines $x = 1$, and $y = 1$ in succession, we obtain that $v(x, y) = cxy(x - 1)(y - 1)$, where $c \in \mathbb{R}$. Finally, $v(x, y)$ vanishes at $(1/2, 1/2)$, so $c = 0$, and thus $v \equiv 0$. Hence, the triple is unisolvent.

unseen ⇓

8, D

2. (a) seen ⇓

$$
\begin{aligned}
D^\beta(Q_{k,B}f)(x) &= D^\beta \frac{1}{|B|}\int_B \sum_{|\alpha|\le k} D^\alpha f(y)\frac{(x-y)^\alpha}{\alpha!}\,\mathrm{d}\,y,\\
&= \frac{1}{|B|}\int_B \sum_{|\alpha|\le k} D^\alpha_y f(y)\frac{(x-y)^\alpha}{(\alpha-\beta)!}\,\mathrm{d}\,y,\\
&= \frac{1}{|B|}\int_B \sum_{|\alpha|\le k} D^\alpha_y f(y) D^\beta_x\frac{(x-y)^\alpha}{\alpha!}\,\mathrm{d}\,y,\\
&= \frac{1}{|B|}\int_B \sum_{|\alpha|\le k} D^\alpha_y f(y)\frac{(x-y)^{\alpha-\beta}}{(\alpha-\beta)!}\,\mathrm{d}\,y,\\
&= \frac{1}{|B|}\int_B \sum_{|\alpha'|\le k-|\beta|} D^{\alpha'+\beta} f(y)\frac{(x-y)^{\alpha'}}{(\alpha')!}\,\mathrm{d}\,y,\\
&= (Q_{k-|\beta|}D^\beta f)(x).
\end{aligned}
$$

8, B

(b) unseen ⇓

$$
\begin{aligned}
\|D^\beta(f-Q_{k,B}f)\|^2_{L^2(K)} &= \|D^\beta f - Q_{k-|\beta|,B}D^\beta f\|^2_{L^2(K)},\\
&\le C|D^\beta f|^2_{H^{k-|\beta|+1}(K)},\\
&\le C\sum_{|\gamma|=|\beta|} |D^\gamma f|^2_{H^{k-|\beta|+1}(K)},\\
&= C\sum_{|\gamma|=|\beta|}\sum_{|\delta|=k-|\beta|+1} |D^{\gamma+\delta} f|^2_{L^2(K)},\\
&= C\sum_{|\gamma|=k+1} |D^\gamma f|^2_{L^2(K)},\\
&= C|f|^2_{H^{k+1}(K)}.
\end{aligned}
$$

8, B

(c) The rescaling arguments involve changing variables from K to K_1 in the integral defining the (squared) seminorm. Since this only involves derivatives of the same degree, we get a common factor of h^{k+1}. If the full norm is used, the derivatives are a mixture of degrees of $k+1$ or less, and the common factor is only h, which spoils the estimate. unseen ⇓

4, D

3. (a) Multiplication by a test function v with $v(0) = v(1) = 0$, and integrate: seen/sim.seen ⇓

$$\int_0^1 v\frac{\mathrm{d}^4 u}{\mathrm{d}\, x^4} - v\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2} + uv \,\mathrm{d}\, x = \int_0^1 fv \,\mathrm{d}\, x.$$

Integration by parts twice in the first term and once in the second:

$$\int_0^1 \frac{\mathrm{d}^2 v}{\mathrm{d}\, x^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2} + \frac{\mathrm{d}\, v}{\mathrm{d}\, x}\frac{\mathrm{d}\, u}{\mathrm{d}\, x} + uv \,\mathrm{d}\, x - \left[v\frac{\mathrm{d}\, u}{\mathrm{d}\, x}\right]_0^1 - \left[v\frac{\mathrm{d}^3 u}{\mathrm{d}\, x^3}\right]_0^1 + \left[\frac{\mathrm{d}\, v}{\mathrm{d}\, x}\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}\right]_0^1 = \int_0^1 fv \,\mathrm{d}\, x.$$

The first two boundary terms vanish since $v(0) = v(1) = 0$, and the third one vanishes due to $\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}(0) = \frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}(1) = 0$. Hence we seek $u \in \mathring{H}^2([0,1])$ such that

$$\int_0^1 \frac{\mathrm{d}^2 v}{\mathrm{d}\, x^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2} + \frac{\mathrm{d}\, v}{\mathrm{d}\, x}\frac{\mathrm{d}\, u}{\mathrm{d}\, x} + uv \,\mathrm{d}\, x = \int_0^1 fv \,\mathrm{d}\, x, \quad \forall v \in \mathring{H}^2([0,1]),$$

where

$$\mathring{H}^2([0,1]) = \{u \in H^2([0,1]) : u(0) = u(1) = 0\}.$$

8, A

(b) One choice is the cubic Hermite element with nodal variables being function values and derivatives at cell endpoints, thus ensuring that the finite element functions are continuous and have continuous derivatives (since these values will agree either side of the cell endpoint). Then $V_h \subset C^1([0,1]) \subset H^2([0,1])$. seen/sim.seen ⇓ 6, A

(c) To satisfy the conditions of the Lax-Milgram theorem, we need to show that $a : V \times V \to \mathbb{R}$ is continuous and coercive in H^2, and $F : V \to \mathbb{R}$ is bounded in H^2, where a and F are the bilinear and linear forms from the left and right hand side of the variational problem, respectively. seen/sim.seen ⇓

$$F[v] \leq \|v\|_{L^2}\|f\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^2},$$

from the Schwarz inequality and the definition of L^2 and H^2 norms, so F is bounded. The bilinear form a is exactly the H^2 inner product, so it has continuity and coercivity constants both equal to 1. Hence the solution exists and is unique, and the required bound holds with $C = 1/1 = 1$. 6, A

4. (a) From the notes, Céa's Lemma says that **sim. seen ⇓**

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma} \min_v \|u - v\|_{H^1},$$

where M is the continuity constant of the bilinear form and γ is the coercivity constant, both using the H^1 norm. For this problem, we have

$$\int_\Omega uv + \alpha \nabla u \cdot \nabla v \, \mathrm{d}\, x \leq \|u\|_{L^2}\|v\|_{L^2} + \max(1,b)|u|_{L^2}|v|_{L^2} \leq (1+\max(1,b))\|u\|_{H^1}\|v\|_{H^1},$$

so an upper bound for the continuity constant is $\max(1,b)$. Similarly, a lower bound for the coercivity constant is $\min(1,a)$. Hence,

$$\|u - u_h\|_{H^1} \leq \frac{\max(1,b)}{\min(1,a)} \min_v \|u - v\|_{H^1}.$$

Choosing $v = \mathcal{I}_h u$, we use the error estimate derived in lectures,

$$\|u - u_h\|_{H^1} \leq \frac{\max(1,b)}{\min(1,a)}\|u - I_h u\|_{H^1} \leq\leq \frac{\max(1,b)}{\min(1,a)} h^2 \|u - I_h u\|_{H^3}.$$

10, B

(b) Following the steps of the Aubin-Nitsche trick, we can choose $g = u - I_h u$. Then,

$$
\begin{aligned}
\|u - I_h u\|_{L^2} &= \langle u - u_h, u - u_h \rangle_{L^2} \\
\text{weak form of } w \text{ equation} &= a(w, u - u_h), \\
\text{Galerkin orthogonality} &= a(w - I_h w, u - u_h), \\
\text{Continuity of } a \quad &\leq M\|u - u_h\|_{H^1}\|w - I_h w\|_{H^1}, \\
\text{error estimate for } w \quad &\leq M\|u - u_h\|_{H^1} Ch|w\|_{H^2}, \\
\text{elliptic regularity for } w \quad &\leq M\|u - u_h\|_{H^1} ChC_1\|u - u_h\|_{L^2}, \\
\text{error estimate for } u \quad &\leq MC_2 h^2 |u|_{H^2} ChC_1 \|u - u_h\|_{L^2},
\end{aligned}
$$

and dividing by $\|u - u_h\|_{L^2}$ gives the result. **10, C**

5. (a) We need to show that B^* is injective. The inf-sup condition says sim. seen ⇓

$$\beta \leq \inf_{0\neq q\in Q} \sup_{0\ neqv\in V} \frac{b(v,q)}{\|v\|_V\|q\|_Q} = \inf_{0\neq q\in Q} \frac{1}{\|q\|_Q} \sup_{0\ neqv\in V} \frac{b(v,q)}{\|v\|_V} = \inf_{0\neq q\in Q} \frac{\|B^*q\|_{V'}}{\|q\|_Q}.$$

Hence, $\|B^*q\|_{V'} \geq \beta\|q\|_Q$ for all $q \in Q$. If there exist q_1, q_2 such that $B^*q_1 = B^*q_2$, then

$$0 = \|B^*(q_1 - q_2)\|_{V'} \geq \beta\|q_1 - q_2\|_Q,$$

so $q_1 = q_2$, i.e. B^* is injective. Therefore B is surjective, so for all G there exists u_g such that $Bu_g = G$. 6, M

(b) The right hand side is continuous as it is the sum of two continuous forms. a is continuous on coercive on V and therefore is continuous and coercive on $Z \subset V$. Hence, Lax-Milgram gives a unique solution u_Z. 6, M

(c) We have $Bu_g = G$ and $Bu_z = 0$ (because $u_z \in Z$). Therefore, $Bu = B(u_g + u_z) = G$, i.e. $b(u,q) = G[q]$ for all $q \in Q$, which is the second of the two equations in the mixed system. Further, if we define

$$L[v] = F[v] - a(u,v), \quad \forall v \in V,$$

we have that

$$L[v] = F[v] - a(u_g,v) - a(u_z,v) = 0, \quad \forall v \in Z.$$

Hence, $L \in (\text{Ker}B)^0$ i.e. $L \in \text{Im}(B^*)$. This means that there exists p such that $B^*p = L$, i.e.

$$b(v,p) = F[v] - a(u,v), \quad \forall v \in V.$$

Rearranging, we have

$$a(u,v) + b(v,p) = F[v], \quad \forall v, \in V,$$

which is the first of the two equations in the mixed system. 8, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 26 of 20 marks

Total C marks: 10 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks