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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH70022 Finite Elements: Numerical Analysis and Implementation

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
Question 1(a-d) 20, Question 2(a) 5, Question 4(a) 5

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
Question 2(b,c(i-ii)) 11, Question 3(a,b) 10,

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
Question 3(c) 5, Question 4(b, c) 10

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):

Question 2(c(iii)) 4, Question 3(d) 5, Question 4(d) 5.

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite Elements: Numerical Analysis and Implementation

Date: ??
Time: ??
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. The 1D Hermite element is defined by $(K, \mathcal{P}, \mathcal{N})$, where:

   1. K is the interval $[a, b]$ (for simplicity in this question we will take $a = 0$, $b = 1$),
   2. $\mathcal{P}$ is the space of polynomials of degree 3 or less,
   3. $N_1(v) = v(a)$, $N_2(v) = v(b)$, $N_3(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(a)$, $N_4(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(b)$.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$. (5 marks)

(b) Show that the polynomial $\phi(x) = 3x^2 - 2x^3$ is one of the nodal basis functions for $\mathcal{P}$. (5 marks)

(c) Which of the following problems is this element suitable for building a Galerkin discretisation for, and why?

$$-u'' = 0, \quad u(0) = 0,\ u'(1) = 1, \tag{1}$$

$$u - u'' + u'''' = \sin(x), \quad u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2}$$

$$\begin{aligned} u - u'' + u'''' - u'''''' = \exp(x), \quad & u(0) = u'(0) = u''(0) = 0, \\ & u(1) = u'(1) = u''(1) = 0. \end{aligned} \tag{3}$$

(5 marks)

(d) Propose a finite element that is suitable for the same problems that you indicated in Part (c), but with $\mathcal{P}$ altered to be the space of polynomials of degree 4 or less. Explain briefly why your proposed finite element is unisolvent. (5 marks)

(Total: 20 marks)

2. Consider the following linear variational problem: find $u \in H^1$ such that

$$a(u, v) = F(v), \quad \forall v \in H^1,$$

where $a(u, v)$ is a bilinear functional with continuity constant M and coercivity constant γ, and $F(v)$ is a bounded linear functional.

For a finite element space $V_h \subset H^1(\Omega)$ defined on a domain Ω, the Galerkin approximation of the linear variational problem is as follows: find $u_h \in V_h$ such that

$$a(u_h, v) = F(v), \quad \forall v \in V_h.$$

(a) Show that

$$a(u - u_h, v) = 0, \quad \forall v \in V_h.$$

(5 marks)

(b) Show that

$$\gamma \|u - u_h\|_{H^1} \leq M \|u - v\|_{H^1},$$

for any $v \in V_h$, where $\|f\|_{H^1}$ is the H^1 norm on Ω. (5 marks)

(c) Consider the following bilinear form on $H^1(\Omega)$,

$$a(u, v) = \int_\Omega uv + \nabla u \cdot \nabla v + v\beta \cdot \nabla u \,\mathrm{d}\, x,$$

where β is a given vector field with $|\beta| \leq \beta_0$, $\nabla \cdot \beta = 0$ and $\beta \cdot n = 0$ on the boundary $\partial\Omega$ of Ω, where n is the unit outward pointing normal to $\partial\Omega$.

(i) Find a finite upper bound for the continuity constant of a. (3 marks)

(ii) Show that

$$a(u, u) = \int_\Omega u^2 + |\nabla u|^2 + \frac{1}{2}\nabla \cdot (\beta u^2) \,\mathrm{d}\, x.$$

(3 marks)

(iii) Hence, find a finite upper bound for the coercivity constant of a. (4 marks)

(Total: 20 marks)

3. (a) For $f \in H^1(\Omega)$, where Ω is some convex polygonal domain, the H^1 projection of f into a degree k Lagrange finite element space V is the function $u \in V$ such that

$$\int_\Omega uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x = \int_\Omega vf + \nabla v \cdot \nabla f \,\mathrm{d}\,x, \quad \forall v \in V.$$

Show that u exists and is unique from this definition, with

$$\|u\|_{H^1} \leq \|f\|_{H^1}.$$

(5 marks)

(b) Show that the H^1 projection is mean-preserving, i.e.

$$\int_\Omega u \,\mathrm{d}\,x = \int_\Omega f \,\mathrm{d}\,x.$$

(5 marks)

(c) Show that the H^1 projection u into V of f is the minimiser over $v \in V$ of the functional

$$J[v] = \int_\Omega (v - f)^2 + \|\nabla(v - f)\|^2 \,\mathrm{d}\,x.$$

(5 marks)

(d) Hence, show that

$$\|u - f\|_{H^1(\Omega)} < Ch^k |f|_{H^{k+1}(\Omega)},$$

where h is the maximum triangle diameter in the triangulation used to construct V. Here, you may quote results from the course without proof.

(5 marks)

(Total: 20 marks)

4. Consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0, \tag{4}$$

solved for a time-dependent function $u(x,t)$ on a closed simply connected domain Ω, with boundary conditions $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$.

(a) Given a C^0 finite element space V_h, formulate a finite element discretisation of the wave equation (4) with time dependent solution $u(x,t) \in V_h$. You may assume that the solution is twice differentiable in time, and should use a variational formulation involving integration over the spatial domain Ω only. (5 marks)

(b) Show that the discretisation can be written in the form

$$M \frac{\mathrm{d}^2}{\mathrm{d}\, t^2} \boldsymbol{u} + K \boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ is the vector of basis coefficients for $u \in V_h$.

(5 marks)

(c) Show that the discretisation is equivalent to the following formulation: simultaneously find $u \in V_h$ and $v \in V_h$ such that

$$\langle \phi, u_t \rangle - \langle \phi, v \rangle = 0, \quad \forall \phi \in V_h, \tag{5}$$

$$\langle \psi, v_t \rangle + \langle \nabla \psi, \nabla u \rangle = 0, \quad \forall \psi \in V_h, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product on Ω. (5 marks)

(d) Using Equations (5-6), show that the solution conserves energy,

$$E = \frac{1}{2} \int_\Omega v^2 + |\nabla u|^2 \, \mathrm{d}\, x,$$

i.e. $\dot{E} = 0$. (5 marks)

(Total: 20 marks)

5. The variational formulation of the Stokes equation seeks $(u, p) \in (V, Q)$, where $V = (\mathring{H}^1)^3$ is the subspace of $(H^1)^3$ vanishing on the boundary, and $Q = \mathring{L}^2$ is the subspace of L^2 that integrates to zero, such that

$$c((u,p),\,(v,q)) = \int_\Omega f \cdot v \,\mathrm{d}\,x, \quad \forall (v,q) \in (\mathring{H}^1(\Omega))^3 \times \mathring{L}^2(\Omega), \tag{7}$$

where

$$c((u,p),\,(v,q)) = a(u,v) + b(v,p) + b(u,q), \tag{8}$$

$$a(u,v) = 2\mu \int_\Omega \epsilon(u) : \epsilon(v) \,\mathrm{d}\,x, \quad b(u,q) = \int_\Omega q \nabla \cdot u \,\mathrm{d}\,x, \tag{9}$$

$\mu > 0$ is the viscosity (a real number), and $\epsilon(v)$ is a mapping from vector fields to tensors given by

$$\epsilon(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right). \tag{10}$$

(a) Show that the bilinear form c is not coercive. (5 marks)

(b) We define the operator $B : V \to Q'$ by

$$Bv[p] = b(v,p), \quad \forall p \in Q,$$

and the transpose operator $B^* : Q \to V'$ by

$$B^* p[v] = b(v,p), \quad \forall v \in V,$$

where V' and Q' are the dual spaces to V and Q respectively.

(i) Show that the inf-sup condition,

$$\inf_{0 \neq q \in Q} \sup_{\neq v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta > 0,$$

is equivalent to

$$\inf_{0 \neq q \in Q} \frac{\|B^* q\|_{V'}}{\|q\|_Q} \geq \beta.$$

(5 marks)

(ii) Hence, show that the inf-sup condition is equivalent to

$$\|B^* q\|_{V'} \geq \beta \|q\|_Q, \quad \forall q \in Q.$$

(5 marks)

(iii) Hence, show that the inf-sup condition implies injectivity of B^*. (5 marks)

(Total: 20 marks)