BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

M70022

Finite Elements (Solutions)

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1. (a) We need to show that for $v \in \mathcal{P}$ then if $N_i(v) = 0$, $i = 1, 2, 3, 4$, then $v \equiv 0$. At $x = 0$ there is a double root because $v(0) = v'(0) = 0$, and similarly at $x = 1$. Therefore v is a degree 3 polynomial with 4 roots, so $v \equiv 0$ by the fundamental theorem of algebra. seen/sim.seen ⇓

5, A

(b) $N_1(\phi) = \phi(0) = 3 \times 0^2 - 2 \times 0^3 = 0$. $N_2(\phi) = \phi'(0) = 6 \times 0 - 6 \times 0^2 = 0$. $N_3(\phi) = \phi(1) = 3 \times 1^2 - 2 \times 1^3 = 1$. $N_4(\phi) = 6 \times 1 - 6 \times 1^2 = 0$. Hence ϕ vanishes for all nodal variables, except for N_3, so it is a nodal basis function.

5, A

(c) The finite element can be used to build a C^1 finite element space, because we have the function value and its derivative at each interval vertex, so we can enforce continuity of the function and its derivative by sharing those nodal variables between cells.

Equation (1) is a second order problem, which requires an H^1 formulation, so the finite element space must be C^0. Our finite element space is $C^1 \subset C^0$, so this is suitable.

Equation (2) is a fourth order problem, which requires an H^2 formulation, so the finite element space must be C^1. Our finite element space is C^1, so this is suitable.

Equation (3) is a sixth order problem, which requies an H^3 formulation, so the finite element space must be C^2. Our finite element space cannot be C^2, because the second derivative at a vertex cannot be computed from nodal variables stored at that vertex, so this is not suitable.

5, A

(d) We need to add an extra nodal variable to those above. For example, $N_5(v) = v((a + b)/2)$. This is unisolvent because if $N_1[v] = 0$ for all $i = 1, 2, 3, 4, 5$, v is a degree 4 polynomial with five roots, hence is the zero polynomial. This is still suitable for problems (1) and (2) because the function value and derivative at each vertex can be obtained from nodal variables associated with that vertex.

5, A

2. (a) Since $V_h \subset H^1$, we can use $v \in V_h$ in both the original variational problem and the Galerkin approximation, [seen ⇓]

$$a(u,v) = F[v],\, a(u_h,v) = F[v].$$

Subtraction and linearity in the first argument gives

$$0 = a(u,v) - a(u_h,v) = a(u-u_h,v),$$

as required. [5, A]

(b) Continuity and coercivity mean that

$$a(u,v) \le M\|u\|_{H^1}\|v\|_{H^1},\, \gamma\|u\|_{H^1}^2 \le a(u,u),$$

respectively. Then, for arbitrary $v \in V_h$, we have

$$\begin{aligned}
\text{(coercivity) } \gamma\|u-u_h\|_{H^1}^2 &\le a(u-u_h,u-u_h),\\
\text{(linearity) } &\le a(u-u_h,u-v) + \underbrace{a(u-u_h,v-u_h)}_{=0 \text{ by part (a)}},\\
\text{(continuity) } &\le M\|u-u_h\|_{H^1}\|u-v\|_{H^1}.
\end{aligned}$$

Either $\|u-u_h\|_{H^1} = 0$, in which case the required result holds as required, or we can divide by $\|u-u_h\|_{H^1}$ to obtain the required result. [5, B]

(c) (i) [sim. seen ⇓]

$$\begin{aligned}
a(u,v) &= \langle u,v\rangle_{H^1} + \int_\Omega v\beta\cdot\nabla u \,\mathrm{d}\,x,\\
&\le \|u\|_{H^1}\|v\|_{H^1} + \|v\|_{L^2}\|\beta\cdot\nabla u\|_{H^1},\\
&\le \|u\|_{H^1}\|v\|_{H^1} + \|v\|_{L^2}\beta_0\|\nabla u\|_{L^2},\\
&\le (1+\beta_0)\|u\|_{H^1}\|v\|^{H^1}.
\end{aligned}$$

The continuity constant is $\le (1+\beta_0)$. [3, B]

(ii) Substituting $v = u$ gives [unseen ⇓]

$$a(u,u) = \int_\Omega u^2 + |\nabla u|^2 + u\beta\cdot\nabla u \,\mathrm{d}\,x.$$

We have $u\nabla u = \frac{1}{2}\nabla u^2$. If $\nabla\cdot\beta = 0$ then $\beta\cdot\nabla\psi = \nabla\cdot(\beta\psi)$ for any scalar field ψ and the result follows by picking $\psi = u^2/2$. [3, B]

(iii)

$$\begin{aligned}
a(u,u) &= \int_\Omega u^2 + |\nabla u|^2 + \frac{1}{2}\nabla\cdot(\beta u^2)\,\mathrm{d}\,x,\\
&= \int_\Omega u^2 + |\nabla u|^2 \,\mathrm{d}\,x + \frac{1}{2}\int_{\partial\Omega}\underbrace{\beta\cdot n}_{=0}\frac{1}{2}u^2\,\mathrm{d}\,S,\\
&= \|u\|_{H^1}^2,
\end{aligned}$$

so the coercivity constant is 1. [4, D]

3. (a) This variational problem has a bilinear form which is just the H^1 inner product. Hence it is continuous and coercive with scaling constants equal to 1. From the Lax-Milgram theorem, the solution exists and is unique. Taking $v = u$, we have

sim. seen ⇓

$$\|u\|_{H^2}^2 = \langle u, f\rangle_{H^2} \leq \|u\|_{H^2}\|f\|_{H^2},$$

from Cauchy-Schwarz, and dividing both sides by $\|u\|_{L^2}$ gives the result.

5, B

(b) Since V is a Lagrange finite element space of degree k, it contains the function $v = 1$. Taking v in the definition gives

$$\int_\Omega u \,\mathrm{d}\, x$$

on the left hand side, and

$$\int_\Omega f \,\mathrm{d}\, x$$

on the right, hence the result.

5, B

(c) Method 1: solve by computing variational derivative,

$$\delta J[v; \delta v] = 2\int_\Omega \delta v(u - f) + \nabla\delta v \cdot \nabla(u - f) \,\mathrm{d}\, x = 0, \quad \forall \delta v \in V,$$

which is equivalent to our variational problem above.
Method 2: by contradiction. If u is not the minimiser, then there exists $v \in V$ with $J[v] \leq J[u]$. Then

$$\begin{aligned} J[v] &= \|v - f\|_{H^1}^2 = \|(v - u) + (u - f)\|_{H^1}^2, \\ &= \|v - u)\|_{H^1}^2 + \underbrace{\langle v - u, u - f\rangle_{H^1}}_{=0 \text{ by defn of } u} + \|u - f\|_{H^1}, \\ &= \|v - u\|_{H^1}^2 + J[u], \end{aligned}$$

and we conclude that $\|v - u\|_{H^1}^2 \leq 0$, a contradiction (since norms cannot be negative and u is assumed not equal to v).

5, C

(d) Since u minimises the functional J, we have

$$\begin{aligned} \|u - f\|_{H^1(\Omega)} &= \inf_{\|v\|_{H^1(\Omega)} > 0} \|v - f\|_{H^1(\Omega)}, \\ &\leq \|I_h f - f\|_{H^1(\Omega)}, \\ &\leq Ch^k |f|_{H^{k+1}(\Omega)}, \end{aligned}$$

where I_h is the nodal interpolation operator into V, and we used the standard approximation result for I_h.

5, D

4. (a) Multiplication by test function and integration by parts in the Laplacian term gives the following variational problem: find (twice time differentiable) time-dependent $u \in H^1$ such that

sim. seen ⇓

$$\langle \psi, u_{tt} \rangle + \langle \nabla \psi, \nabla u \rangle = 0, \quad \forall \psi \in H^1.$$

Our C^0 finite element space is a subset of H^1 so we may propose the following finite element discretisation, find $u_h \in V_h$ such that

$$\langle \psi, u_{h,tt} \rangle + \langle \nabla \psi, \nabla u_h \rangle = 0, \quad \forall \psi \in V_h.$$

5, A

(b) Using a basis for V_h of dimension N, we substitute basis expansions for ψ and u, leading to

$$\sum_i^N \psi_i \left(\sum_j^N \int_\Omega \phi_i \phi_j \, \mathrm{d}\, x \frac{\mathrm{d}^2}{\mathrm{d}\, t^2} u_j + \sum_j^N \nabla \phi_i \cdot \nabla \phi_j \, \mathrm{d}\, x u_j w \right) = 0.$$

Since the basis coefficients ψ_i are arbitrary, we must have

$$\sum_j^N \underbrace{\int_\Omega \phi_i \phi_j \, \mathrm{d}\, x}_{=M_{ij}} \frac{\mathrm{d}^2}{\mathrm{d}\, t^2} u_j + \sum_j^N \underbrace{\nabla \phi_i \cdot \nabla \phi_j \, \mathrm{d}\, x}_{=K_{ij}} u_j = 0, \quad i = 1, \ldots, N,$$

which is equivalent to the required form.

5, C

(c) If we introduce $v \in V_h$ such that

unseen ⇓

$$\langle \phi, u_t \rangle - \langle \phi, v \rangle = 0, \quad \forall \phi \in V_h,$$

(which is Equation (5)) then choosing $\phi = u_t - v \in V_h$ gives

$$0 = \langle u_t - v, u_t - v \rangle = \|u_t - v\|_{L^2}^2 \implies u_t = v.$$

Hence, $u_{tt} = v_t$, and we can substitute into the variational form to get Equation (6), and the two formulations are equivalent.

5, C

(d)

$$\begin{aligned} \dot{E} &= \langle v, v_t \rangle + \langle \nabla u, \nabla u_t \rangle, \\ (u_t = v) \quad &= \langle v, v_t \rangle + \langle \nabla u, \nabla v \rangle, \\ (\text{Equation 6}) &= -\langle \nabla v, \nabla u \rangle + \langle \nabla u, \nabla v \rangle = 0, \end{aligned}$$

as required.

5, D

5. (a) Taking $v = 0$, $p \neq 0$, we have

$$c((0,p),(0,p)) = a(0,0) + b(0,p) + b(0,p) = 0,$$

by bilinearity (or the definition). Hence we have a pair (v,p) with $\|v\|_V^2 + \|p\|_Q^2 > 0$, but $c((v,p),(v,p)) = 0$, i.e. c is not coercive. 5, M

(b) (i) We have

$$\|B^*p\|_{V'} = \sup_{0 \neq v \in V} \frac{B^*p[v]}{\|v\|_V} = \sup_{0 \neq v \in V} \frac{b(v,p)}{\|v\|_V},$$

and hence

$$\inf_{0 \neq p \in Q} \frac{\|B^*p\|_{V'}}{\|p\|_Q} = \inf_{0 \neq p \in Q} \frac{b(v,p)}{\|v\|_V \|p\|_Q},$$

so the two conditions are equivalent. 5, M

(ii) For any q,

$$\frac{\|B^*q\|_{V'}}{\|q\|_Q} \geq \inf_{0 \neq p \in Q} \frac{\|B^*p\|_{V'}}{\|p\|_Q} \geq \beta,$$

by the definition of $\inf$, so

$$\|B^*q\|_{V'} \geq \beta \|q\|_Q,$$

for any q. Starting from this end, we take $q \neq 0$, divide by $\|q\|_Q$, and $\inf$ over all such q, to recover the original expression. 5, M

(iii) If there exists q_1, q_2 such that $B^*q_1 = B^*q_2$, and then $B^*(q_1 - q_2) = 0$ by linearity. Then Part b(ii) says that

$$0 = \|B^*(q_1 - q_2)\|_{V'} \geq \beta \|q_1 - q_2\|_Q,$$

i.e. $q_1 = q_2$, so B^* is injective. 5, M

Review of mark distribution:

Total A marks: 30 of 32 marks
Total B marks: 21 of 20 marks
Total C marks: 15 of 12 marks
Total D marks: 14 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks