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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2023

MATH60022/70022 Finite Elements: Numerical Analysis and Implementation

The following information must be completed:

Is the paper suitable for resitting students from previous years:
Yes for all 3rd year students
Yes for 4th year/MSc students who took the course in 2021-2022 and 2020-2021
No for 4th year/Msc students from previous years (different topic for mastery question)

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 8 marks; 2(a) 8 marks; 1(a) 3 marks; 4(a) 10 marks

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(b) 5 marks; 2(b) 5 marks; 3(b) 5 marks; 4(b) 6 marks

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
1(c) 3 marks; 2(c) 3 marks; 3(c) 3 marks; 4(c) 4 marks

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(d) 4 marks; 2(d) 4 marks; 3(d) 4 marks

Signatures are required for the final version:

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite Elements: Numerical Analysis and Implementation

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH60022 paper; 2.5 Hours for MATH70022 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Let $(K, P, \mathcal{N})$ be a finite element.

   (i) Provide a definition of the nodal variables $\mathcal{N}$. (3 marks)

   (ii) What does it mean for $\mathcal{N}$ to determine $\mathcal{P}$? (3 marks)

(b) Let $(K, P, \mathcal{N})$ be defined by:

* K is a triangle with vertices z_1, z_2 and z_3.
* P is $P_1 + \mathring{P}_3$, where P_1 are the linear polynomials, and $\mathring{P}_3$ is the subspace of the cubic polynomials P_3 that vanish on the boundary of K,
* $\mathcal{N} = (N_1, N_2, N_3, N_4)$, where $N_i[u] = u(z_i)$, $i = 1, 2, 3$, and $N_4[u] = u(z^*)$, where $z^* = z_1 + (z_2 - z_1)/3 + (z_3 - z_1)/3$.

   (i) Let $u \in P$. Show that u is linear when restricted to each of the edges of K. (1 mark)

   (ii) Show that $\mathcal{N}$ determines P. (4 marks)

(c) Find the nodal basis function ϕ_4. (3 marks)

(d) Let u solve $-\nabla^2 u = f$ in the unit square Ω with boundary conditions $u = 0$ on all sides of the square.
Let V be the finite element space constructed on a triangulation $\mathcal{T}$ of the unit square, using the finite element considered in Part (b).
Let $u_h \in \mathring{V}$ solve the finite element variational problem,

$$\int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}\,x = \int_\Omega f v \,\mathrm{d}\,x, \quad \forall v \in \mathring{V}, \tag{1}$$

where $\mathring{V}$ is the subspace of V containing functions that vanish on the boundary $\partial\Omega$ of Ω.
Let $b \in V$ be a function that vanishes everywhere except for one triangle K in $\mathcal{T}$. Further, let $b = \phi_4$ in K.

   (i) Show that

$$\int_K \nabla(u - u_h) \cdot \nabla b \,\mathrm{d}\,x = 0. \tag{2}$$

(3 marks)

   (ii) Hence, show that

$$\int_K u - u_h \,\mathrm{d}\,x = \int_{\partial K} (u - u_h)\gamma(x) \,\mathrm{d}\,S, \tag{3}$$

where ∂K is the boundary of K and $\gamma(x)$ is a known function supported on ∂K. Give the definition of $\gamma(x)$. (3 marks)

(Total: 20 marks)

2. (a) Consider a finite element $(K, P, \mathcal{N})$, with nodal basis $\{\phi_i\}_{i=1}^n$.

(i) Provide a definition of local interpolation operator I_K. (2 marks)

(ii) Show that

$$N_i[I_K(v)] = N_i[v], \quad i = 1, \ldots, n. \tag{4}$$

(3 marks)

(iii) Show that I_K is the identity when restricted to P. (3 marks)

(b) For a nodal variable $N \in P'$, we define the norm $\|N\|_{C^l(K)'}$ by

$$\|N\|_{C^l(K)'} = \sup_{0<\|u\|_{C^l(K)}} \frac{|N[u]|}{\|u\|_{C^l(K)}}, \tag{5}$$

where

$$\|u\|_{C^l(K)} = \sup_{x\in K, r=0,\ldots,l} |D_r u(x)|, \tag{6}$$

and D_r is the rth derivative. Show that

$$\|I_K(u)\|_{H^k(K)} \leq C_1 \|u\|_{C^l(K)}, \tag{7}$$

where

$$C_1 = \sum_{i=1}^n \|\phi_i\|_{H^k(K_1)} \|N_i\|_{C^l(K)'}. \tag{8}$$

(5 marks)

(c) You may assume the Sobolev inequality for continuous functions $u \in C^l(K)$. This states that there exists a constant $0 < C_2 \leq \infty$, depending only on the shape and size of K, such that

$$\|u\|_{C^l(K)} \leq C_2 \|u\|_{H^k(K)}, \tag{9}$$

provided that $k > d/2 + l$, where d is the dimension of K.
Show that there exists $0 < C_3 \leq \infty$, such that

$$\|I_K(u)\|_{H^k(K)} \leq C_3 \|u\|_{H^k(K)}, \tag{10}$$

stating any assumptions that you make about k, l, and d. (3 marks)

(d) For each finite element below, consider $u \in H^2(K)$, and give all the values of k for which $\|I_K(u)\|_{H^k(K)}$ is finite, justifying your answer.

(i) The degree m Lagrange elements on the interval $[0, 1]$. (1 mark)

(ii) The degree m Lagrange elements on a triangle. (1 mark)

(iii) The degree m Lagrange elements on a tetrahedron. (1 mark)

(iv) The cubic Hermite elements (nodal variables are point evaluations on vertices plus cell centre, and derivative evaluations on vertices) on a triangle. (1 mark)

(Total: 20 marks)

3. (a) Let $(K, P, \mathcal{N})$ be a finite element.

   (i) Provide a definition of a local geometric decomposition for $(K, P, \mathcal{N})$. (4 marks)

   (ii) What does it mean for a local geometric decomposition to be C^0? (4 marks)

(b) Let $(K, P, \mathcal{N})$ be a finite element with a C^0 local geometric decomposition. Let $\mathcal{T}$ be a triangulation, and let V be a finite element space constructed on $\mathcal{T}$ using $(K, P, \mathcal{N})$ and the local geometric decomposition. Show that V is a C^0 finite element space. (5 marks)

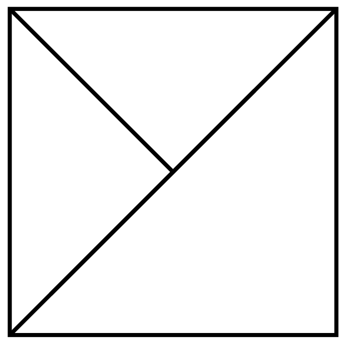

Figure 1: Diagram for parts 3(c-d)

(c) The diagram in Figure 1 shows a mesh which is not a triangulation. We can nevertheless proceed to define a finite element space V on this mesh, using linear Lagrange elements with the C^0 geometric decomposition assigning each nodal variable to its vertex.

   (i) Show that V is not a C^0 finite element space. (1 mark)

   (ii) Describe, with justification, a subspace of V containing only C^0 functions. (2 marks)

(d) We consider the finite element discretisation of the equation $u - \nabla^2 u = f$ on the unit square Ω with boundary conditions $\frac{\partial u}{\partial n} = 0$ on all edges. For a C^0 finite element space W, the finite element discretisation seeks $u_h \in W$ such that

$$\int_\Omega u_h v + \nabla u_h \cdot \nabla v \, \mathrm{d}\, x = \int_\Omega v f \, \mathrm{d}\, x, \quad \forall v \in W. \tag{11}$$

We say that a finite element discretisation is *consistent* if replacing u_h with the exact solution u to the strong form partial differential equation still gives equality for all test functions. Provide a modification to (11) with the two following properties:

1. The resulting finite element discretisation is consistent.
2. The modification vanishes when u_h vanishes on the interior edge going from bottom left to top right in the diagram in Figure 1.

(4 marks)

(Total: 20 marks)

4. For a convex polygonal domain Ω, we consider a variational problem on $H^1(\Omega)$, seeking $u \in H^1(\Omega)$ such that

$$a(u, v) = F[v], \quad \forall v \in H^1(\Omega), \tag{12}$$

where a and F are bilinear and linear forms on $H^1(\Omega)$, respectively. Further, we assume that a is symmetric.

(a) (i) State what it means for $a(\cdot, \cdot)$ to be coercive. (2 marks)

(ii) State what it means for $a(\cdot, \cdot)$ to be continuous. (2 marks)

(iii) State what it means for F to be continuous. (2 marks)

(iv) Write down a Galerkin finite element approximation to our variational problem, using a finite element space $V \subset H^1(\Omega)$. (4 marks)

(b) (i) Show that

$$a(u - u_h, v) = 0, \quad \forall v \in V, \tag{13}$$

where u_h is the finite element approximation to the solution u. (3 marks)

(ii) What does this tell us about the error in the solution? (3 marks)

(c) Using the earlier parts of this question, and assuming that a is continuous and coercive, show that

$$\|u - u_h\|_{H^1(\Omega)} \leq C \sup_{v \in V} \|v - u\|_{H^1(\Omega)}, \tag{14}$$

for some constant $0 < C \leq \infty$. (4 marks)

(Total: 20 marks)

5. In this question, we consider the Stokes equations, written in strong form as

$$-2\mu\nabla\cdot\epsilon(u)+\nabla p=f,\quad \nabla\cdot u=0, \tag{15}$$

for (vector valued) velocity u and (scalar valued) pressure p, where

$$\epsilon(u)=\frac{1}{2}\left(\nabla u+(\nabla u)^T\right). \tag{16}$$

We consider boundary conditions $u=0$ on the boundary $\partial\Omega$ of the 3-dimensional domain Ω.

The variational formulation seeks $(u,p)\in(V,Q)$, where $V=(\mathring{H}^1)^3$ is the subspace of $(H^1)^3$ vanishing on the boundary, and $Q=\mathring{L}^2$ is the subspace of L^2 that integrates to zero, such that

$$c((u,p),\,(v,q))=\int_\Omega f\cdot v\,\mathrm{d}\,x,\quad \forall(v,q)\in(\mathring{H}^1(\Omega))^3\times\mathring{L}^2(\Omega), \tag{17}$$

where

$$c((u,p),\,(v,q))=a(u,v)+b(v,p)+b(u,q), \tag{18}$$

$$a(u,v)=2\mu\int_\Omega\epsilon(u):\epsilon(v)\,\mathrm{d}\,x,\quad b(u,q)=\int_\Omega pq\,\mathrm{d}\,x. \tag{19}$$

(a) Show that if (u,p) solves the variational formulation, and further that $u\in H^2(\Omega)$ and $p\in H^1(\Omega)$, then (u,p) solves the strong form of the Stokes equations.

(8 marks)

(b) Show that the form $c((u,p),(v,q))$ is not coercive. (3 marks)

(c) Now we consider the discrete inf-sup condition, which requires that there exists $\beta_h>0$ such that

$$\inf_{0\neq q\in Q_h}\sup_{0\neq v\in V_h}\frac{b(v,q)}{\|v\|_V\|q\|_Q}\geq\beta_h, \tag{20}$$

for finite element spaces $V_h\subset V$ and $Q_h\subset Q$. We define B^* as the map from Q to the dual space V', given by

$$(B^*q)[v]=b(v,p),\quad \forall q\in Q,\,v\in V. \tag{21}$$

We also define B_h^* as the map from Q_h to the dual space V_h', given by

$$(B_h^*q)[v]=b(v,p),\quad \forall q\in Q_h,\,v\in V_h. \tag{22}$$

Show that if B_h^* has a kernel, then the discrete inf-sup condition is not satisfied. (4 marks)

(d) Consider a mesh of squares, with each square subdivided into four triangles by the diagonals. When V_h is constructed from continuous linear elements (with the boundary condition subspace restriction) and Q_h is constructed from discontinuous constant elements (with the mean zero restriction), show that $\ker(B_h^*)$ is not empty. (5 marks)

(Total: 20 marks)