BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

M70022

Finite Elements (Solutions)

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1. (a) seen ⇓

(i) The nodal variables $\mathcal{N}$ are a basis for the dual space P' of P. $\boxed{3,\ \mathrm{A}}$

(ii) $\mathcal{N}$ determines P if $\mathcal{N}$ is indeed a basis for P'. $\boxed{3,\ \mathrm{A}}$

(b) unseen ⇓

(i) If $v = P_1 + \mathring{P}_3$ then $v = v_1 + \mathring{v}_3$ where $v_1 \in P_1$ and $\mathring{v}_3 \in \mathring{P}_3$. On an edge of K, $\mathring{v}_3 = 0$, so $v = v_1$ there, *i.e* v is linear when restricted to the edge. $\boxed{1,\ \mathrm{B}}$

(ii) seen/sim.seen ⇓

We define lines Π_1, Π_2, Π_3 intersecting z_1, z_2, z_2, z_3, and z_3, z_1 respectively. We choose nondegenerate linear functionals L_1, L_2, L_3 that vanish on Π_1, Π_2, Π_3 respectively. Now assume that $v \in P$ is such that $N_i[v] = 0$ for $i = 1, 2, 3, 4$. v restricted to Π_1 vanishes at two points, z_1 and z_2, so $v = 0$ on Π_1 by the fundamental theorem of algebra. Thus $v = L_1 q_1$ for a quadratic polynomial q_1 (since v is cubic). Similarly, $v = 0$ on Π_2. Therefore q_1 vanishes everywhere on Π_2 except potentially at z_2, but continuity requires that q_1 vanishes there too. Hence, $q_1 = L_2 q_3$, where q_3 is linear. Similarly, $v = 0$ on Π_3, hence $q_3 = cL_3$ for c constant. Finally, v vanishes at z^*, but none of L_1, L_2, L_3 vanish there, so we must have $c = 0$, i.e. $v \equiv 0$. $\boxed{4,\ \mathrm{B}}$

(c) unseen ⇓

ϕ_4 vanishes on Π_1, Π_2, Π_3, so by similar arguments to above, $\phi_4 = cL_1L_2L_3$. We need $\phi_4(z^*) = 1$, so
$\phi_4(x) = L_1(x)L_2(x)L_3(x)/(L_1(z^*)L_2(z^*)L_3(z^*))$. $\boxed{3,\ \mathrm{C}}$

(d) unseen ⇓

(i) u solves the same variational problem, but with $\mathring{V}$ replaced by $\mathring{H}^1$. Taking $v = b$ in both variational problems and subtracting gives

$$0 = \int_\Omega \nabla(u - u_h) \cdot \nabla b \, \mathrm{d}\, x = \int_K \nabla(u - u_h) \cdot \nabla b \, \mathrm{d}\, x, \tag{1}$$

since b is only supported in K. $\boxed{3,\ \mathrm{D}}$

(ii) Integration by parts gives

$$-\int_K (u - u_h)\nabla^2 b \, \mathrm{d}\, x + \int_{\partial K} (u - u_h)\nabla b \cdot n \, \mathrm{d}\, S = 0. \tag{2}$$

b is cubic, so $\nabla^2 b$ is constant and nonzero, we can divide by $c_0 = \nabla^2 b$ to obtain the result, with $\gamma(x) = \nabla b \cdot n / c_0$. $\boxed{3,\ \mathrm{D}}$

2. (a) seen ⇓

(i)

$$I_K[u](x) = \sum_{i=1}^{n} N_i[u]\phi_i(x). \tag{3}$$

2, A

(ii)

$$N_i[I_K[u]] = N_i[\sum_{j=1}^{n} N_j[u]\phi_j(x)] = \sum_{j=1}^{n} N_j[u]N_i[\phi_j(x)]] = \sum_{j=1}^{n} N_j[u]\delta_{ij} = N_i[u], \tag{4}$$

using linearity of N_i and the definition of the nodal basis. 3, A

(iii) If $v \in P$, then we can write $v = \sum_i v_i\phi_i$. Then,

$$I_K(v) = \sum_i v_i I_K(\phi_i) = \sum_i v_i \sum_j N_j[\phi_i]\phi_j = \sum_i v_i \sum_j \delta_{ji}\phi_j = \sum_i v_i\phi_i = v, \tag{5}$$

as required. 3, A

(b) seen ⇓

$$\|I_K(u)\|_{H^k(K)} = \left\| \sum_{i=1}^{n} N_i[u]\phi_i(x) \right\|_{H^k(K)} \tag{6}$$

$$\text{triangle inequality } \leq \sum_{i=1} |N_i[u]| \|\phi_i\|_{H^k(K)} \tag{7}$$

$$\text{definition of } C^l(K)' \text{ norm } \leq \underbrace{\sum_i \|\phi_i\|_{H^k(K)} \|N_i\|_{C^l(K)'}}_{=C_1} \|u\|_{C^l(K)}, \tag{8}$$

as required. 5, B

(c) If $k > d/2 + l$, then we can use the Sobolev inequality to get

$$\|I_K(u)\|_{H^k(K)} \leq C_1 \|u\|_{C^l(K)} \leq \underbrace{C_1 C_2}_{=C_3} \|u\|_{H^k(K)}. \tag{9}$$

3, C

(d) (i) We have $l = 0$ because Lagrange elements involve function evaluation only. We have $d = 1$ because we are solving on an interval. Hence, we need $k > 1/2$. On the other hand, we only have $u \in H^2$, so we can take $k = 1$ or $k = 2$. unseen ⇓ 1, D

(ii) We have $l = 0$ for Lagrange, and $d = 2$, so we need $k > 1$. This means that only $k = 2$ is possible. 1, D

(iii) We have $l = 0$ for Lagrange, and $d = 3$, so we need $k > 3/2$. This means that only $k = 2$ is possible. 1, D

(iv) We have $l = 1$ for Hermite, and $d = 2$, so we need $k > 2$. This means that no values of k are possible. 1, D

3. (a) (i) A local geometric decomposition for $(K, P, \mathcal{N})$ is an assignment of each nodal variable $N \in \mathcal{N}$ to a geometric entity of K. seen $\Downarrow$

4, A

(ii) A local geometric decomposition for $(K, P, \mathcal{N})$ is C^0, if for each geometric entity w of K, there exists a subset $\mathcal{N}_w \subset \mathcal{N}$ containing only nodal variables that have been assigned to the closure of w, such that $(w, P|_w, \mathcal{N}_w)$ is a finite element, where $P|_w$ is the restriction of P to w. 4, A

(b) We need to show that $u \in V$ means that $u \in C^0$. To do this we need to check continuity of u across vertices, edges, and in 3D, faces. If V is constructed using elements with a C^0 geometric decomposition, then we can take any global entity of the triangulation (i.e., a vertex, edge, face, or cell), and the value of u should agree on w from any cell that contains w. If $(w, P|_w, \mathcal{N}_w)$ is a finite element, then since u in each cell shares those nodal variables, the value of u is completely determined on w in the same way from all cells. seen $\Downarrow$ 5, B

(c) (i) If we take the function which is zero on each vertex of the square, but equal to one in the middle, then there is a discontinuity because the function is zero in the entire bottom right triangle, but one in the middle. unseen $\Downarrow$ 1, C

(ii) The subspace requires that the value in the middle is the average of the values at the bottom left and top right vertices. Then, the function is linear along the entire diagonal, which matches the values in the bottom right triangle. 2, C

(d) Denote the diagonal edge as Γ. A consistent modification is to add a term unseen $\Downarrow$

$$-\int_{\Gamma} n^+ \cdot \nabla u_h^+ v_h^+ + n^- \cdot \nabla u_h^- v_h^- \, \mathrm{d}\, S \tag{10}$$

to the left hand side, where $+$ and $-$ indicate the values above and below Γ respectively. This is consistent since if we replace u_h with the exact solution u, we can separately integrate by parts in the regions above and below Γ, to obtain

$$\int_{\Omega} (u - \nabla^2 u - f) v \, \mathrm{d}\, x = 0, \tag{11}$$

as required. The modification vanishes according to property 2 because it only involves values of u_h on that boundary. 4, D

4. (a) (i) a is coercive if there exists a constant $\gamma > 0$ such that seen ⇓

$$a(u,u) \geq \gamma \|u\|_{H^1}^2, \quad \forall u \in H^1. \tag{12}$$

2, A

(ii) a is continuous if there exists a constant $C > 0$ such that

$$a(u,v) \leq C\|u\|_{H^1}\|v\|_{H^1}, \quad \forall u, v \in H^1. \tag{13}$$

2, A

(iii) F is continuous if there exists a constant $C > 0$ such that

$$F[v] \leq C\|v\|_{H^1}, \quad \forall v \in H^1. \tag{14}$$

2, A

(iv) The Galerkin approximation seeks $u_h \in V$ such that

$$a(u_h, v) = F[v], \quad \forall v \in V. \tag{15}$$

4, A

(b) (i) We take $v \in V$ in the H^1 variational problem (possible since $V \subset H^1$, and substract the Galerkin approximation with the same v, using linearity, seen ⇓

$$a(u - u_h, v) = a(u,v) - a(u_h, v) = F[v] - F[v] = 0, \tag{16}$$

as required. 3, B

(ii) The error is $u - u_h$. This tells us that the error is orthogonal to the whole of V, when using $a(\cdot,\cdot)$ as an inner product. seen ⇓

(c) For arbitrary $v \in V$, 3, B

seen ⇓

$$\begin{aligned}
\gamma\|u - u_h\|_{H^1(\Omega)}^2 &\leq a(u - u_h, u - u_h) && (17)\\
&= a(u - u_h, u - v) + \underbrace{a(u - u_h, v - u_h)}_{=0} && (18)\\
&\leq C\|u - u_h\|_{H^1(\Omega)}\|u - v\|_{H^1(\Omega)}, && (19)
\end{aligned}$$

and the result is obtained by dividing by $\|u - u_h\|_{H^1(\Omega)}$ and sup-ing over all $v \in V$.

4, C

5. (a) We first note that if u solves the weak form equation, then taking $q = \nabla\cdot u$, $v = 0$ gives $\|\nabla\cdot u\|_{L^2} = 0$, i,e. $\nabla\cdot u = 0$ in L^2. If $u \in H^2$ and $p \in H^1$, we may integrate by parts to get

$$\int_\Omega (-\mu\nabla^2 u - \mu\nabla \underbrace{(\nabla\cdot u)}_{=0} + \nabla p - f) v \,\mathrm{d}\, x, \quad \forall v \in V, \tag{20}$$

having dropped the boundary integral because v vanishes there. Then, we may choose as v a sequence of C_0^∞ functions converging to $-\mu\nabla^2 u + \nabla p - f$, and hence conclude that $-\mu\nabla^2 u + \nabla p = f$ in L^2.

seen/sim.seen ⇓

8, M

(b) If we take $u = 0$, then

$$c((u,p),(u,p)) = 0, \tag{21}$$

so c is not coercive.

seen ⇓

3, M

(c) We have

$$\|B_h^* q\|_{V_h'} = \sup_{0\neq v\in V_h} \frac{b(v,q)}{\|v\|_V}, \tag{22}$$

so the inf sup condition is equivalent to

$$\inf_{0\neq q\in Q_h} \|B^* h q\|_{V_h'} \geq \gamma > 0. \tag{23}$$

If B_h^* has a kernel, then we can take q in the kernel and get zero, violating the inf sup condition.

unseen ⇓

4, M

(d) We consider a function $q \in Q_h$ that is only supported in one square (subdivided into triangles). Inside the square, q is either -1 or 1, with the value alternating upon crossing the diagonal lines between triangles. We claim that $b(u,q) = 0$ for all $u \in V_h$. To check this, we just need to check it for each basis function supported in the square. The basis function equal to 1 at the square centre has constant divergence, so the q values cancel out and $b(u,q) = 0$. A basis function equal to 1 at a corner of the square also has constant divergence inside its support, and the same thing happens. Therefore, $q \in \ker B_h^*$. In fact there is one kernel function for each square.

unseen ⇓

5, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 21 of 20 marks

Total C marks: 13 of 12 marks

Total D marks: 14 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks