Module:	MATH97095
Setter:	Cotter
Checker:	Ham
Editor:	Wu
External:	external
Date:	March 6, 2023
Version:	Draft Version

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2021

MATH97095 Finite Elements

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes, but only from 2020/21, because in years before that the mastery content was different.

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1a:7, 1b:7, 1c:6, 2a:6, 2b:6 [Total 32]

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
4a:6, 4b:6, 2c:8 [Total 20]

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
3a:6, 3c:6 [Total 12]

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
4c:8, 3b:8 [Total 16]

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite Elements

Date: Wednesday, 5th May 2021

Time: 09:30 – 11:30

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Consider the following finite element

- K is a triangle,
- P is the space of polynomials of degree ≤ 2,
- N is the set of six nodal variables given by evaluation at the vertices and edge centres of K.

(a) Show that N determines P. (7 marks)

(b) Give a C^0 geometric decomposition of this finite element, showing that it is C^0. (7 marks)

(c) Show that finite element spaces built from this element are not necessarily C^1. (6 marks)

(Total: 20 marks)

2. (a) Write a C^0 finite element variational problem for the following equation,

$$\epsilon u - \nabla^2 u = \exp(xy), \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\Omega = \{x, y : 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$ with boundary $\partial\Omega$, and $0 < \epsilon < 1$.

(6 marks)

(b) Show that the bilinear form for the variational problem is continuous and coercive, and give bounds for the continuity and coercivity constants M and γ for this problem.
(You may make use of the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$.)

(6 marks)

(c) Assuming Ceá's Lemma and standard interpolation error estimates, derive an error bound for the H^1 error $\|u - u_h\|$ where u_h is the solution obtained by a linear Lagrange finite element approximation with maximum mesh size h, and u is the exact solution. What is happening to this error bound when ϵ is very small?

(8 marks)

(Total: 20 marks)

3. In this question we consider the following finite element.

- K is a triangle.
- P are vector-valued linear functions (i.e. the x- and y- components of the function are both polynomials of degree ≤ 1).
- The six nodal variables are the components of the function tangential to the edges at the locations indicated by the arrows in Figure ??.

(a) Describe how this element can be used to construct a finite element space V where the functions are continuous in the tangential component across each edge. Show that the finite element space does indeed have this property. (6 marks)

(b) Consider the quadratic Lagrange finite element space P_2. Show that $\phi \in P_2 \implies \nabla\phi \in V$. (8 marks)

(c) Provide a formula for the weak curl $\nabla^{\perp}\cdot u = -\partial u_1/\partial y + \partial u_2/\partial x$ for a function $u = (u_1, u_2) \in V$, and show that it is indeed the weak curl. (6 marks)

(Total: 20 marks)

4. Let Ω be a convex polygonal domain. Assume that you have a fast and efficient code for solving the variational problem: find $u \in V$ such that

$$\int_\Omega uv + \nabla u \cdot \nabla v \,\mathrm{d}\, x = F[v], \quad \forall v \in V, \tag{2}$$

for arbitrary linear functionals $F[v]$, where V is a C^0 finite element space. However, you want to solve a different variational problem: find $u \in V$ such that

$$\int_\Omega a(x)uv + b(x)\nabla u \cdot \nabla v \,\mathrm{d}\, x = G[v], \quad \forall v \in V, \tag{3}$$

where $a(x)$ and $b(x)$ are some known functions that satisfy $0 < \alpha < a(x) < \beta < \infty$, $0 < \alpha < b(x) < \beta < \infty$, for all $x \in \Omega$. One possible approach is to apply the following iterative scheme,

$$\int_\Omega u^{k+1}v + \nabla u^{k+1} \cdot \nabla v \,\mathrm{d}\, x = F_k[v], \tag{4}$$

where

$$F_k[v] = \int_\Omega u^k v + \nabla u^k \cdot \nabla v \,\mathrm{d}\, x + \mu \left(G[v] - \int_\Omega a(x)u^k v + b(x)\nabla u^k \cdot \nabla v \,\mathrm{d}\, x \right), \quad \forall v \in V, \tag{5}$$

where $\mu > 0$, for an iterative sequence $u^0, u^1, u^2, \ldots$ of guesses at the solution. To implement this, we choose an initial guess u^0, and then iteratively generate the sequence by solving (4) for u^{k+1} given u^k (which enables us to construct $F_k[v]$).

(a) Show that if the sequence converges to a limit $u_k \to u^*$ as $k \to \infty$, then u^* solves Equation (3). (6 marks)

(b) Defining the error $\epsilon^k = u - u^k$, where u solves (3), derive a variational problem that relates ϵ^{k+1} to ϵ^k (without explicitly involving u^{k+1} or u^k). (6 marks)

(c) Find a value of μ such that $\|\epsilon^{k+1}\|_{H^1} < \|\epsilon^k\|_{H^1}$, concluding that the iterative procedure converges. You may make use of the stability bound from Lax-Milgram, i.e. the solution u to a variational problem satisfies

$$\|u\|_{H^1} \le \frac{1}{\gamma}\|F\|_{(H^1)^*}, \tag{6}$$

where γ is the coercivity constant of the bilinear form and F is the linear form appearing on the right hand side. (8 marks)

(Total: 20 marks)

5. (a) Let V and Q be Hilbert spaces. Let $b : V \times Q \to \mathbb{R}$ be a bilinear form. We define the operator $B : V \to Q'$ as follows. For each $v \in V$, Bv is an element of Q', defined by

$$(Bv)[p] = b(v,p),\ \forall p \in Q. \tag{7}$$

For an operator $T : X \to Y'$, we define the transpose operator $X^* : Y \to X'$ as

$$(T^*y)[x] = (Tx)[y], \quad \forall x \in X,\ y \in Y. \tag{8}$$

Use these definitions to derive a formula for B^*. (6 marks)

(b) Assuming the inf-sup condition

$$\inf_{0 \neq q \in Q} \sup_{0 \neq v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta, \tag{9}$$

for some $\beta > 0$, show that B^* is injective. (7 marks)

(c) Let

$$b(u,p) = \int_\Omega p \nabla \cdot u \,\mathrm{d}\, x, \tag{10}$$

for some chosen problem domain Ω such that b satisfies the inf-sup condition for some given finite element spaces V_h and Q_h. We define the "weak gradient" operator $\tilde{\nabla} : Q_h \to V_h$ such that

$$\int_\Omega w \cdot \tilde{\nabla} p \,\mathrm{d}\, x = \int_\Omega p \nabla \cdot w \,\mathrm{d}\, x, \qquad \forall w \in V_h \tag{11}$$

What does the inf-sup condition imply about the operator $\tilde{\nabla}$? (7 marks)

(Total: 20 marks)