1. (a) Consider the finite element $(K, P, \mathcal{N})$ given by

1. K is the 1×1 square, with bottom-left corner at $(0, 0)$.
2. P is the polynomial space spanned by $\{1, x, y, xy\}$.
3. $\mathcal{N} = (N_1, N_2, N_3, N_4)$ where $N_i(p) = p(z_i)$ and (z_1, z_2, z_3, z_4) are the four corners of the square.

Find the nodal basis for this finite element. (You may use a computational linear algebra package such as Numpy or Matlab to invert matrices but you must write the matrices that you are computing with in your solution.) (8 marks)

(b) Consider a finite element $(K, P, \mathcal{N})$ with

1. K is the triangle with vertices at $z_1 = (0, 0)$, $z_2 = (1, 0)$, and $z_3 = (0, 1)$.
2. P is the polynomial space spanned by $\{1, x, y, xy(1 - x - y)\}$,

and nodal basis

$$\begin{gathered}\psi_1 = x - 9xy(1 - x - y),\ \psi_2 = y - 9xy(1 - x - y),\\ \psi_3 = 1 - x - y - 9xy(1 - x - y),\ \psi_4 = 27xy(1 - x - y).\end{gathered} \tag{1}$$

(i) Find a set of nodal variables $\mathcal{N}$ that corresponds to this nodal basis, justifying your answer. (6 marks)

(ii) Provide a C^0 geometric decomposition for this element, explaining why it has the specified continuity. (6 marks)

(Total: 20 marks)

2. In this question we consider the Poisson equation

$$-\nabla^2 u = f, \tag{2}$$

on a convex domain Ω with $u = 0$ on $\partial\Omega$. We assume that f is such that $u \in H^2(\Omega)$ but $u \notin H^3(\Omega)$.

(a) Consider Theorem 5.30 of the notes. Explain why this theorem is not sufficient for estimating the convergence rate for finite element discretisation for this problem when $k = 2$.

(6 marks)

(b) For $i = 1$, propose and prove a modification of Lemma 5.28 for this case.

(7 marks)

(c) For $i = 1$, propose and justify (referring to existing proofs in the notes) a modification of Theorem 5.30 for this case. Comment on the difference in the estimate compared to the case $u \in H^3(\Omega)$. (7 marks)

(Total: 20 marks)

3. (a) Write a finite element variational problem for the following equation,

$$-\nabla^2 u = 0, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega, \tag{3}$$

describing the types of finite element spaces that should be used to ensure a unique solution.

(6 marks)

(b) Consider the finite element discretisation in the case where Ω is a 1×1 square and $g = \exp(\cos(x)\cos(y+x))$. Explain why your variational problem is not possible to implement exactly on a computer when using numerical quadrature, and propose a modification that is.

(6 marks)

(c) Adjust the statement and proof of Céa's Lemma to accommodate your modification from (b), and comment on the conditions for convergence of the finite element solution to the exact solution as the mesh is refined. (Hints: start from the triangle inequality for $\|u - v + v - u_h\|_{H^1(\Omega)}$ for $v \in V_h$ and then work with the term $\|v - u_h\|_{H^1(\Omega)}$.)

(8 marks)

(Total: 20 marks)

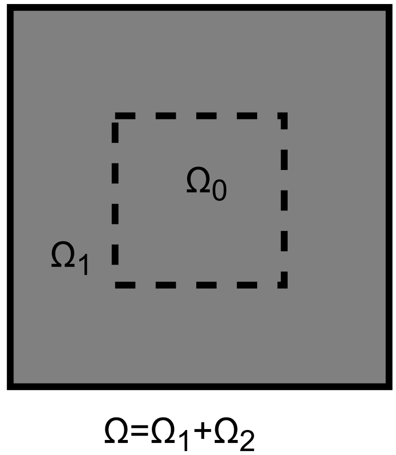

Figure 1: Domain for Question 4.

4. In this question we consider the domain displayed in Figure 1. The domain Ω is the entire square area shaded grey, with outer boundary $\partial\Omega$ shown as a continuous black line. Inside the domain is a smaller square, denoted Ω_0, with boundary Γ, shown as a dashed black line. We define Ω_1 to be the complement of Ω_0 in Ω.

We consider the following problem: find u such that

$$-\nabla^2 u = 0, \text{ in } \Omega_0 \text{ and } \Omega_1, \tag{4}$$

$$u = 0, \text{ on } \partial\Omega, \tag{5}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_0} + \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_1\cap\Gamma} = 2, \tag{6}$$

where for a domain Ω_i, $\frac{\partial u}{\partial n}|_{\Omega_i}$ is the value of the normal component of the derivative restricted to Ω_i, using the outward pointing normal to $\partial\Omega_i$. Note that since the outward pointing normals to $\partial\Omega_0$ and $\partial\Omega_1 \cap \Gamma$ are equal and opposite, this condition on Γ indicates a discontinuity in the normal derivative of u.

(a) Using the continuous Lagrange finite element space of degree k, formulate a finite element discretisation for this problem. (6 marks)

(b) Show that the finite element discretisation has a unique solution, and provide a constant γ such that the finite element solution satisfies

$$\|u_h\|_{H^1(\Omega)} \leq \gamma, \tag{7}$$

independently of h. You may quote results from the course notes without proof. (8 marks)

(c) Let u_h be the numerical solution on a mesh consisting of squares subdivided into right angled triangles, with square edge length h, and let u be the exact solution of the problem. Discuss the applicability of the bound on $\|u_h - u\|_{H^1(\Omega)}$ that we studied in the course. (6 marks)

(Total: 20 marks)

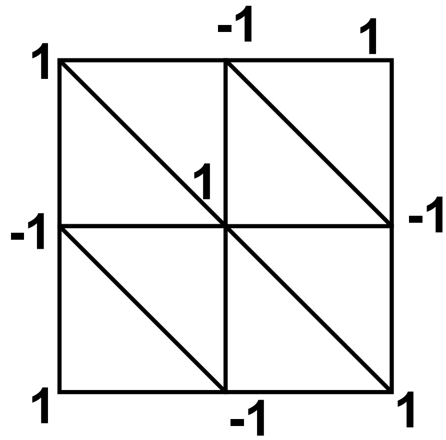

Figure 2: Example function values for Question 5.

5. (a) Let $V = (H^1(\Omega))^2$ and $Q = \mathring{L}^2(\Omega)$, where Ω is a convex polygonal domain. Let $b : V \times Q \to \mathbb{R}$ be the bilinear form

$$b(v, q) = \int_\Omega q \nabla \cdot v \, \mathrm{d}\, x. \tag{8}$$

Assuming the result that the divergence is surjective from V to Q, show that b satisfies the inf-sup condition

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta, \tag{9}$$

for some constant β.

(5 marks)

(b) Define the operator $\delta : Q \to V$ by

$$\int_\Omega w \cdot \delta q \, \mathrm{d}\, x = b(w, q), \quad \forall w \in V. \tag{10}$$

Show that the kernel of δ, $\mathrm{Ker}(\delta)$, is empty. (Hint: find a suitable choice of test function.)

(5 marks)

(c) Let $V_h \subset V$ and $Q_h \subset Q$ be finite element spaces chosen for the discretisation of Stokes' equation. Let $\Pi_h : V \to V_h$ satisfy condition 1 of Fortin's trick, i.e.

$$b(v - \Pi_h v, q) = 0, \quad \forall v \in V,\, q \in Q_h. \tag{11}$$

Define the discrete operator $\delta_h : Q_h \to V_h$ by

$$\int_\Omega w \cdot \delta_h q \, \mathrm{d}\, x = b(w, q), \quad \forall w \in V_h. \tag{12}$$

Show that $\mathrm{Ker}(\delta_h) \subseteq \mathrm{Ker}(\delta)$.

(5 marks)

(Question continues on the following page.)

(d) Consider a mesh consisting of squares subdividing into right angle triangles by joining the top left vertex and the bottom right vertex of each square, and consider the discretisation for Stokes with continuous linear Lagrange elements for each component of the velocity and continuous linear Lagrange elements for the pressure.

By considering the function $p \in Q_h$ taking vertex values in an alternating pattern as indicated in Figure 2, show that $\mathrm{Ker}(\delta_h) \not\subseteq \mathrm{Ker}(\delta)$ in this case.
Do V_h and Q_j satisfy condition 1 of Fortin's trick? (5 marks)

(Total: 20 marks)