1. (a) The Vandermonde matrix and its inverse are

sim. seen ⇓

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

which can be computed by hand by e.g. doing back-substitution on the columns of the identity matrix. The basis is then

$$\psi_1(x,y) = 1 - x - y + xy = (1-x)(1-y), \tag{2}$$
$$\psi_2(x,y) = x - xy = x(1-y), \tag{3}$$
$$\psi_3(x,y) = y - xy = y(1-x), \tag{4}$$
$$\psi_4(x,y) = xy. \tag{5}$$

8, A

(b) (i) Suitable nodal variables are $N_i(p) = p(z_i)$ where $z_1 = (1,0)$, $z_2 = (0,1)$, $z_3 = (0,0)$, $z_4(1/3,1/3)$.

unseen ⇓

We can check that the basis is a nodal one for these nodes by noticing that the spanning set for $P$ is the linear functions plus a cubic "bubble" function that vanishes on the triangle vertices (and the edges). Thus to make a nodal basis for our nodal variables, the basis functions 1 to 3 can just be the usual linear basis functions (that are equal to 1 on the corresponding vertex and 0 at all the others) plus a scalar multiple of the bubble function so that they vanish at $z_4$. The bubble vanishes at all the vertices so it just needs to be scaled appropriately to take the value 1 at $z_4$ as required.

6, A

(ii) We assign $N_i$ to vertex $z_i$ for $i = 1, 2, 3$, and the bubble function the entire cell. This is a C0 geometric decomposition because: (1) the value at each vertex can be obtained from the nodal variable assigned to that vertex (since it is just point evaluation at the vertex), (2) the value at each edge can be obtained from the nodal variables assigned to the closure of the edge, which is just vertex values at each end in this case, and the function is linear when restricted to an edge.

sim. seen ⇓

6, B

2. (a) The theorem is insufficient because $|u|_{H^3(K_1)}$ is unbounded, so it doesn't provide any bound on the error. unseen ⇓

6, B

(b) The appropriate statement is (under the same conditions as 5.28 but with $u \in H^2(K_1)$, $k = 3$), sim. seen ⇓

$$|\mathcal{I}_{K_1} u - u|_{H^1(K_1)} \leq C_1 |u|_{H^2(K_1)}. \tag{6}$$

To prove it,

$$|\mathcal{I}_{K_1} u - u|_{H^1(K_1)} \leq \|Q_{3,B} u - u\|^2_{H^1(K_1)} + \|\mathcal{I}_{K_1}(u - Q_{3,B} u)\|^2_{H^1(K_1)} \tag{7}$$

$$\leq (1 + C^2)|u|^2_{H^2(K_1)}, \tag{8}$$

where $Q_{3,B}$ is the degree $k$ averaged Taylor polynomial over a ball $B$ inside $K_1$ but as large as possible, and where we used Lemmas 3.22 and Corollary 3.16.

7, B

(c) The appropriate statement is (under the same conditions as 5.30 but with $u \in H^2(\Omega)$)

$$|\mathcal{I}_K u - u|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}. \tag{9}$$

To show this, note that we can obtain the local estimate

$$|\mathcal{I}_K u - u|_{H^1(K)} \leq C_K d|u|_{H^{k+1}(K)}, \tag{10}$$

by following the steps in the proof but with $k$ replaced by 1. Then the same technique of summing over all the cells gives the global result.

7, C

3. (a) To derive the variational form, we multiply by a test function $v$ and integrate by parts as usual to get

sim. seen ⇓

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x - \int_{\partial\Omega} v \underbrace{\frac{\partial u}{\partial n}}_{=g} \mathrm{d}\, S = 0, \tag{11}$$

so a suitable variational form is to find $v \in \bar{V}_h$ such that

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x = \int_{\partial\Omega} vg \, \mathrm{d}\, S, \quad \forall v \in \bar{V}_h, \tag{12}$$

where $\bar{V}_h$ is the subspace of $V_h$ of functions that integrate to zero, and $V_h$ is some choice of $C^0$ finite element space.

6, A

(b) The issue is that the integrals are not tractable in general, so we can't evaluate the RHS of the problem. A possible modification is to interpolate $g$ to $V_h$ in the boundary resulting in $g_h$, and solve the perturbed problem

unseen ⇓

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x = \int_{\partial\Omega} vg_h \, \mathrm{d}\, S, \quad \forall v \in \bar{V}_h. \tag{13}$$

6, D

(c) The modification to Céa's Lemma is

unseen ⇓

$$\|u - u_h\|_{H^1(\Omega)} \leq (1 + M/\gamma) \sup_{v \in V_h} \|u - v\|_{H^1(\Omega)} + \frac{C}{\gamma}\|g - g_h\|_{L^2(\partial\Omega)}, \tag{14}$$

so there are now two terms, a best approximation term of $u$ in $V_h$, and an approximation error term for $g_h$.

To prove it, following the steps of Céa's Lemma, we take a test function $v \in V_h$ in both the exact and approximate equation, and compute the difference, to yield

$$a(u - u_h, v) = \int_{\partial\Omega} v(g - g_h) \, \mathrm{d}\, S, \, \forall v \in V_h. \tag{15}$$

Then we use coercivity to write (for any $v \in V_h$)

$$\begin{aligned}
\gamma\|u_h - v\|_{H^1(\Omega)} &\leq a(u_h - v, u_h - v), && (16)\\
&= a(u_h - u, u_h - v) + a(u - v, u_h - v), && (17)\\
&= \int_{\partial\Omega} (u_h - v)(g_h - g) \, \mathrm{d}\, S + a(u - v, u_h - v), && (18)\\
&\leq C\|u_h - v\|_{H^1(\Omega)}\|g_h - g\|_{L^2(\partial\Omega)} + M\|u - v\|_{H^1(\Omega)}\|u_h - v\|_{H^1(\Omega)}, && (19)
\end{aligned}$$

where $C$ is the constant in the trace inequality and $M$ is the continuity constant of the bilinear form $a(u, v)$. Then, dividing by $\|u_h - v\|_{H^1(\Omega)}$ gives

$$\gamma\|u_h - v\|^2_{H^1(\Omega)} \leq C\|g_h - g\|_{L^2(\partial\Omega)} + M\|u - v\|_{H^1(\Omega)}. \tag{20}$$

Then, combining with the triangle inequality, we get

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \|u - v\|_{H^1(\Omega)} + \|u_h - v\|_{H^1(\Omega)}, && (21)\\
&\leq (1 + M/\gamma)\|u - v\|_{H^1(\Omega)} + \frac{C}{\gamma}\|g - g_h\|_{L^2(\partial\Omega)}, && (22)
\end{aligned}$$

and minimisation over $v$ gives the result.

8, D

4. (a) Multiplying by a test function $v$ that vanishes on the exterior boundary and integrating by parts separately in $\Omega_1$ and $\Omega_2$ gives

unseen ⇓

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x - \int_\Gamma v \left( \frac{\partial u}{\partial n}|_{\partial \Omega_1} + \frac{\partial u}{\partial n}|_{\partial \Omega_2 \cap \Gamma} \right) \mathrm{d}\, S = 0, \tag{23}$$

and substitution of the boundary condition gives the variational problem: find $u_h \in V_h$ such that

$$\int_\Omega \nabla v \cdot \nabla u_h \, \mathrm{d}\, x = 2 \int_\Gamma v \, \mathrm{d}\, S, \quad \forall v \in V_h. \tag{24}$$

6, A

(b) We are in the case of Theorem 4.38, so we just need to check continuity of the linear form according to

sim. seen ⇓

$$F[v] = 2 \int_\Gamma v \, \mathrm{d}\, S \leq \|v\|_{L^2(\Gamma)} 2|\Gamma| \leq \|v\|_{H^1(\Omega_0)} 2|\Gamma| \leq \|v\|_{H^1(\Omega)} 2|\Gamma|. \tag{25}$$

where we have used the trace theorem for continuous finite elements (Theorem 4.4), and $|\Gamma| = \int_\Gamma \mathrm{d}\, S$. Hence $F$ is continuous.

8, A

(c) The bound studied in the course is

unseen ⇓

$$\|u_h - u\|_{H^1(\Omega)} \leq h|u|_{H^2(\Omega)}, \tag{26}$$

but the solution has a jump in the first derivative across $\Gamma$, so $|u|_{H^2(\Omega)}$ is not finite, so the bound does not imply convergence of the numerical solution as $h \to 0$.

6, C

5. (a) The map is surjective, so there exists $v \in V$ such that $q = \nabla \cdot v$ for all $q \in Q$. Hence, using the Riesz Representation Theorem, for all $F \in Q'$, there exists $q_F$ such that unseen ⇓

$$F[p] = \int_\Omega p q_F \,\mathrm{d}\,x, \quad \forall p \in Q. \tag{27}$$

So, for all $F \in Q'$, there exists $v \in V$ such that

$$b(v, p) = \int_\Omega \nabla \cdot v p \,\mathrm{d}\,x = F[p], \quad \forall p \in Q. \tag{28}$$

In other words, for all $F \in Q'$ there exists $v$ such that $Bv = F$, which means that $B$ is surjective. Then, from the notes, this implies the inf-sup condition. 5, M

(b) Let $q \in \mathrm{Ker}(\delta)$, i.e. $\delta q = 0$. Taking $w$ such that $\nabla \cdot w = q$, we have unseen ⇓

$$0 = \int_\Omega \nabla \cdot w q \,\mathrm{d}\,x = \int_\Omega q^2 \,\mathrm{d}\,x \implies q = 0. \tag{29}$$

5, M

(c) Let $q \in \mathrm{Ker}(\delta_h)$. Then for $w \in V$, unseen ⇓

$$\int_\Omega w \cdot \delta q \,\mathrm{d}\,x = b(w, q) = b(\Pi_h w, q), \tag{30}$$

$$= \int_\Omega \Pi_h w \cdot \delta_h q \,\mathrm{d}\,x = 0, \tag{31}$$

so $q \in \mathrm{Ker}(\delta)$ as required. 5, M

(d) Considering $p \in Q_h$ having a pattern of the type given in Figure 2, it suffices to consider $b(w, p)$ for $w$ being basis functions associated with vertices in the interior of the mesh, which span $V_h$ (because of the zero boundary condition). The support of $w$ consists of 6 triangles with symmetry about a diagonal line from top-left to bottom-right. The divergence of $w$ is antisymmetric about that line, whilst $p$ is symmetric, so the integral $b(w, p)$ vanishes. Hence, $p \in \ker(\delta_h)$. The previous result says that the Fortin Trick assumptions imply that the $\ker(\delta_h)$ is empty, $V_h, Q_h$ must fail to satisfy the Fortin Trick assumptions. unseen ⇓ 5, M