1. This question is about the equation

$$c\phi + \boldsymbol{u}\cdot\nabla\phi - \epsilon\nabla^2\phi = f \text{ on } \Omega, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where:

- Ω is a d-dimensional polygonal domain with boundary $\partial\Omega$,
- $c > 0$,
- f is a known function,
- $\boldsymbol{u} \in C^{1,\infty}(\Omega)^d$ is a known vector-valued function satisfying $\nabla\cdot\boldsymbol{u} = 0$, and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$.
- $|\boldsymbol{u}|_\infty = \max_{\boldsymbol{x}\in\Omega}|\boldsymbol{u}(\boldsymbol{x})| = C_0 < \infty$.

(a) Derive a weak formulation of this equation for a solution $\phi \in H^1(\Omega)$ of the form

$$a(q,\phi) = F(\phi), \quad \forall H^1(\Omega). \tag{2}$$

[5 marks]

Solution: ***UNSEEN***

Multiplying by a test function q and integrating by parts, we obtain

$$\int_\Omega cq\phi - \nabla\cdot(\boldsymbol{u}q)\phi + \epsilon\nabla q\cdot\nabla\phi \,\mathrm{d}\,x = \int_\Omega fq \,\mathrm{d}\,x, \quad \forall q \in H^1(\Omega).$$

Equal credit for the formulation without integrating by parts in the advection term (they are equivalent).

(b) Obtain estimates for the continuity and coercivity constants of $a(\cdot,\cdot)$.

[10 marks]

Solution: ***UNSEEN***

From Cauchy-Schwarz,

$$\begin{aligned}|a(q,\phi)| &= \int_\Omega cq\phi - q\phi\nabla\cdot\boldsymbol{u} - \phi\boldsymbol{u}\cdot\nabla q + \epsilon\nabla q\cdot\nabla\phi \,\mathrm{d}\,x,\\ &\le c\|q\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} + C_0\|\nabla q\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} + \epsilon\|\nabla q\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)},\\ &\le (c + C_0 + \epsilon)\,\|q\|_{H^1(\Omega)}\|\phi\|_{H^1(\Omega)},\end{aligned}$$

so the continuity constant is $c + C_0 + \epsilon$.

To compute the coercivity constant, first note that the advection term is skew-symmetric, since

$$\begin{aligned}\int_\Omega \nabla\cdot(\boldsymbol{u}q)\phi\,\mathrm{d}\,x &= \int_\Omega \underbrace{\nabla\cdot\boldsymbol{u}}_{=0}q\phi + (\boldsymbol{u}\cdot\nabla q)\phi\,\mathrm{d}\,x,\\ &= -\int_\Omega \nabla\cdot(\boldsymbol{u}\phi)q\,\mathrm{d}\,x,\end{aligned}$$

after integrating by parts. Hence,

$$\begin{aligned}a(\phi,\phi) &= \int_\Omega c\phi^2 + \epsilon|\nabla\phi|^2\,\mathrm{d}\,x,\\ &> \min(c,\epsilon)\|\phi\|_{H^1(\Omega)},\end{aligned}$$

so the coercivity constant is $\min(c,\epsilon)$.

(c) What happens to the H^1 norm of the error in the P^1 finite element approximation of this problem as $\epsilon \to 0$? Justify your answer.

[5 marks]

Solution: ***UNSEEN***

From Céa's Lemma, we have

$$\|\phi_h - \phi\|_{H^1(\Omega)} \leq \frac{c + C_0 + \epsilon}{\epsilon} \|\phi\|^2_{H^1(\Omega)},$$

(assuming that $c > \epsilon$), which tends to infinity as $\epsilon \to 0$. Hence the error can be arbitrarily large in that limit.

2. (a) For a ball B in a triangle K, the averaged Taylor polynomial of a function $u \in H^k(K)$ of degree k is defined by

$$Q_{k,B}u(x) = \frac{1}{|B|}\int_B \sum_{|\alpha|\le k} D^\alpha u(y)\frac{(x-y)^\alpha}{\alpha!}\,\mathrm{d}\,y. \tag{3}$$

For $|\beta| \le k$ show that

$$D^\beta Q_{k,B}u(\boldsymbol{x}) = Q_{k-|\beta|,B}D^\beta u(\boldsymbol{x}). \tag{4}$$

[8 marks]

Solution: ***SEEN***

For continuous functions $u \in C^k(K)$, we have

$$T_{k,y}u(x) = \sum_{|\alpha|\le k} D^\alpha u(y)\frac{(x-y)^\alpha}{\alpha!}.$$

Then,

$$\begin{aligned}
D^\beta T_{k,y}u(x) &= D^\beta \sum_{|\alpha|\le k} D^\alpha u(y)\frac{(x-y)^\alpha}{\alpha!},\\
&= \sum_{|\beta|\le|\alpha|\le k} D^\alpha u(y)\frac{(x-y)^{\alpha-\beta}}{(\alpha-\beta)!},\\
&= \sum_{|\alpha|\le k-|\beta|} D^{\alpha+\beta} u(y)\frac{(x-y)^{\alpha}}{(\alpha)!},\\
&= \sum_{|\alpha|\le k-|\beta|} D^{\alpha} D^\beta u(y)\frac{(x-y)^{\alpha}}{(\alpha)!},
\end{aligned}$$

$$= T_{k-|\beta|y}D^\beta u(x).$$

Then,

$$\begin{aligned}
D^\beta Q_{k,B}u(x) &= D^\beta \frac{1}{|\beta|}\int_B T_{k,y}u\,\mathrm{d}\,x,\\
&= \frac{1}{|\beta|}\int_B T_{k-|\beta|,y}D^\beta u\,\mathrm{d}\,x,\\
&= Q_{k-|\beta|,B}D^\beta u(x).
\end{aligned}$$

(b) For the rest of the question we assume that K has radius 1. Let $u \in H^{k+1}(K)$. Assuming that, for $i \le k$,

$$\|Q_{i,B}u - u\|_{L^2(K)} \le C|u|_{H^{k+1}(K)}, \tag{5}$$

show that

$$\|D^\beta(Q_{i,B}u - u)\|_{L^2(K)} \le C|u|_{H^{k+1}(K)}, \tag{6}$$

for $|\beta| \le i \le k$.

[8 marks]

Solution: ***SEEN***

From the previous result

$$D^\beta(Q_{i,B}u - u) = Q_{i-|\beta|,B}D^\beta u - D^\beta u,$$

so

$$\|D^\beta(Q_{i,B}u - u)\|_{L^2(K)} = \|Q_{i-|\beta|,B}D^\beta u - D^\beta u\|_{L^2(K)} \leq C|D^\beta u|_{H^{k-|\beta|+1}(K)} = C|u|_{H^{k+1}(K)}.$$

(c) Using the property

$$\|I_K u\|_{H^k(K)} \leq C_1\|u\|_{H^k(K)}, \tag{7}$$

for the nodal interpolation operator I_K corresponding to a finite element $(K, \mathcal{P}, \mathcal{N})$, show that

$$|I_K u - u|_{H^k(K)} \leq C_2|u|_{H^{k+1}(K)}, \tag{8}$$

for some positive constant C_2, stating any assumptions you make about $(K, \mathcal{P}, \mathcal{N})$.

[4 marks]

Solution: ***SEEN***

$$\begin{aligned}
|I_k u - u|_{H^k(K)} &\leq |I_k u - Q_{k,B}u + Q_{k,B}u - u|_{H^k(K)}, \\
&\leq |I_k u - Q_{k,B}u|_{H^k(K)} + |Q_{k,B}u - u|_{H^k(K)}, \\
&= |I_k(u - Q_{k,B}u)|_{H^k(K)} + |Q_{k,B}u - u|_{H^k(K)}, \\
&\leq |I_k(u - Q_{k,B}u)|_{H^k(K)} + |Q_{k,B}u - u|_{H^k(K)}, \\
&\leq (C_1 + 1)|Q_{k,B}u - u|_{H^k(K)}, \\
&\leq (C_1 + 1)C_2|u|_{H^{k+1}(K)},
\end{aligned}$$

as required, where we used that $\mathcal{P}$ contains all polynomials of degree k in the third line. In the last line we used the result of part (b) with $|\beta| = i = k$, with $C_2 = CN$ where N is the number of multi-indices β with $|\beta| = k$. So, $C_2 \leq (C_1 + 1)C_2$.

3. Consider the following triple $(K, \mathcal{P}, \mathcal{N})$.

- K is a triangle with vertices z_1, z_2, z_3.
- $\mathcal{P}$ are the polynomials of degree ≤ 3.
- $\mathcal{N}$ are dual variables given by evaluations at $z_1 + (z_2 - z_1)i/3 + (z_3 - z_1)j/3$ for $0 \leq i \leq j \leq 3$.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$.

[10 marks]

Solution: ***SEEN SIMILAR***

Let $p \in \mathcal{P}$ be mapped to zero by all dual variables. Restricted to Π_1, the line between z_2 and z_1, p is a degree 3 polynomial vanishing at 4 places, so it is zero. Therefore $p = L_1(x)Q_1(x)$ where $L_1(x)$ is a non-degenerate linear function vanishing on Π_1. Working through the other two lines, we obtain that $p = cL_1(x)L_2(x)L_3(x)$, where none of the L functions vanish away from the three edges of the triangle. But p also vanishes at the centre of the triangle, so $c = 0$ i.e. $p = 0$ as required.

(b) Describe the geometric decomposition for this finite element, and explain why it is a C^0 decomposition.

[10 marks]

Solution: ***SEEN SIMILAR***

We associate the point evaluation at vertices to their corresponding edges, point evaluation on edges away from vertices to their corresponding edges, and point evaluation at the centre to the triangle itself. Being C^0 requires that when restricted to a vertex, the function can be reconstructed purely from the corresponding vertex node. This is clear because there is just one point value to reconstruct. Being C^0 also requires that when restricted to an edge, the function can be reconstructed purely from nodal variables assigned to the closure of the edge. Since we have 4 values along the edge including the two vertices, this completely determines the cubic function along that edge.

4. Consider the heat equation,

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T, \tag{9}$$

solved for a time-dependent function T on a closed simply-connected domain Ω, with boundary conditions $\frac{\partial T}{\partial n} = 0$ on the boundary $\partial\Omega$.

(a) Given a C^0 finite element space, formulate a finite element discretisation of the heat equation (9).

[5 marks]

Solution: ***UNSEEN***

The variational form is obtained by multiplying both sides by a test function v and integrating by parts to obtain

$$\langle v, T_t \rangle_{L^2} = -\kappa \langle \nabla v, \nabla T \rangle_{L^2}.$$

The finite element discretisation is then find a time-dependent $T \in V_h$ such that

$$\langle v, T_t \rangle_{L^2} = -\kappa \langle \nabla v, \nabla T \rangle_{L^2}, \quad \forall v \in V_h.$$

(b) Show that the discretisation can be written in the form

$$M\dot{\boldsymbol{T}} = K\boldsymbol{T}, \tag{10}$$

where $\boldsymbol{T}$ is the vector of basis coefficients for T in the finite element space V_h.

[5 marks]

Solution: ***UNSEEN***

Introducing basis expansions

$$v = \sum_i v_i \phi_i(x), \quad T = \sum_i T_i(t)\phi(x),$$

we get

$$\sum_i v_i \sum_j \left(\langle \phi_i, \phi_j \rangle_{L^2} \dot{T}_j + \kappa \langle \nabla \phi_i, \nabla \phi_j \rangle_{L^2} T_j \right) = 0,$$

but this equation must hold for arbitrary basis coefficients, so

$$\sum_j \left(\underbrace{\langle \phi_i, \phi_j \rangle_{L^2}}_{=M_{ij}} \dot{T}_j + \kappa \underbrace{\langle \nabla \phi_i, \nabla \phi_j \rangle_{L^2}}_{=K_{ij}} T_j \right) = 0,$$

as required.

(c) Quoting results from lectures, show that

$$\frac{d}{dt} \int_\Omega T^2 \, \mathrm{d}\, x \leq -C \int_\Omega T^2 \, \mathrm{d}\, x, \tag{11}$$

providing an upper bound for the decay rate C.

[5 marks]

Solution: ***UNSEEN***

For Dirichlet boundary conditions we have the result

$$\int_\Omega T^2 \,\mathrm{d}\,x \leq C_p \int_\Omega |\nabla T|^2 \,\mathrm{d}\,x.$$

Then, taking $v = T$ in the variational form,

$$\begin{aligned}\frac{d}{dt}\int_\Omega T^2 \,\mathrm{d}\,x &= 2\int_\Omega T T_t \,\mathrm{d}\,x, \\ &= -2\kappa \int_\Omega |\nabla T|^2 \,\mathrm{d}\,x, \\ &\leq -\frac{2\kappa}{C_p}\int_\Omega T^2 \,\mathrm{d}\,x.\end{aligned}$$

(d) Explain why this means that the decay rate for the finite element discretisation is larger than or equal to the decay rate for the unapproximated equation.

[5 marks]

Solution: ***UNSEEN***

The Poincaré constant for $\mathring{H}^1$ is

$$C_p^* = \sup_{T\in \mathring{H}^1(\Omega)} \frac{\int_\Omega T^2 \,\mathrm{d}\,x}{\int_\Omega |\nabla T|^2 \,\mathrm{d}\,x}.$$

The Poincaré constant for $\mathring{V}_h \subset \mathring{H}^1$ is

$$C_p^h = \sup_{T\in \mathring{V}_h} \frac{\int_\Omega T^2 \,\mathrm{d}\,x}{\int_\Omega |\nabla T|^2 \,\mathrm{d}\,x} \leq \sup_{T\in \mathring{H}^1} \frac{\int_\Omega |T|^2 \,\mathrm{d}\,x}{\int_\Omega |\nabla T|^2 \,\mathrm{d}\,x} = C_p^*,$$

so $C_p^h \leq C_p^*$.

5. This question is based upon the Mastery material "From Functional Analysis to Iterative Methods" by RC Kirby.
Consider the partial differential equation

$$-\nabla \cdot (\gamma(x)\nabla u) = f, \tag{12}$$

on Ω, with boundary conditions $u = 0$ on $\partial\Omega$, where f is a known function with $\|f\|_{L^2(\Omega)} < \infty$, and γ is a known function with $c_1 \leq \gamma \leq c_2$ for $c_1 > 0$, $c_2 < \infty$.

(a) Briefly formulate a finite element discretisation for this problem using linear continuous finite elements, and explain how the coercivity and continuity constants of the variational problem depend on c_1 and c_2. Give details on the function spaces involved and norms involved.

[6 marks]

Solution: ***SEEN SIMILAR***

The weak form is

$$\int_\Omega \gamma \nabla v \cdot \nabla u \, \mathrm{d}\, x = \int_\Omega v f \, \mathrm{d}\, x,$$

with the variational problem defined on $\mathring{H}^1(\Omega)$, the subspace of $H^1(\Omega)$ with traces satisfying the zero Dirichlet boundary condition. Then the finite element discretisation has the same weak form with $\mathring{V} \subset \mathring{H}^1(\Omega)$.

The bilinear form is continuous since

$$\int_\Omega \gamma \nabla v \cdot \nabla u \, \mathrm{d}\, x \leq b \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},$$

so the continuity constant is b, and coercive since

$$\int_\Omega \gamma |\nabla u|^2 \, \mathrm{d}\, x \geq a |u|^2_{H^1(\Omega)} \geq a(1 + C) \|u\|^2_{H^1(\Omega)},$$

where C is constant for the Poincarè inequality, so the coercivity constant is $a(1 + C)$.

(b) A bilinear form a on a finite element space V_h defines an operator $A_h : V_h \to V_h'$ into the dual space given by

$$(A_h f)[u] = a(f, u), \quad \forall f, u \in V_h. \tag{13}$$

In the notation of the paper, the operator $\mathcal{I}_h : \mathbb{R}^{\dim V_h} \to V_h$ maps a vector to the function in V_h with the vector entries as basis coefficients in the nodal basis expansion. The operator $\mathcal{I}_h' : \mathbb{R}^{\dim V_h} \to V_h'$ maps vectors to linear functionals $F \in V_h'$ given by

$$(\mathcal{I}_h' \boldsymbol{f})[u] = \boldsymbol{f}^T (\mathcal{I}_h^{-1} u), \quad \forall u \in V_h. \tag{14}$$

(i) Show that

$$A_h u = \mathcal{I}_h'(A\boldsymbol{u}), \quad \forall u \in V_h. \tag{15}$$

where A is the matrix corresponding to A_h and $\boldsymbol{u}$ is the vector of basis coefficients of u.

[4 marks]

Solution: ***SEEN***

Let $v = \mathcal{I}_h \boldsymbol{v}$. *Then*

$$\begin{aligned}\langle A_h u, v\rangle &= a(u, v),\\ &= a\left(\sum_i u_i\phi_i, \sum_j v_j\phi_j\right),\\ &= \sum_{i,j} u_i v_j a(\phi_i, \phi_j),\\ &= \sum_{i,j} u_i v_j A_{ij},\\ &= (A\boldsymbol{u})^T\boldsymbol{v},\\ &= (A\boldsymbol{u})^T(\mathcal{I}_h^{-1}\mathcal{I}_h\boldsymbol{v}),\\ &= \langle \mathcal{I}_h' A\boldsymbol{u}, v\rangle, \quad \forall v \in V_h,\end{aligned}$$

as required.

(ii) Hence show that

$$A = (\mathcal{I}_h')^{-1} A_h \mathcal{I}_h. \tag{16}$$

[3 marks]

Solution: ***SEEN***

We have $\mathcal{I}_h\boldsymbol{u} = u$, *hence*

$$A\boldsymbol{u} = (\mathcal{I}')_h^{-1} A_h u = (\mathcal{I}')_h^{-1} A_h \mathcal{I}_h \boldsymbol{u}, \quad \forall \boldsymbol{u} \in V_h,$$

as required.

(c) Now consider a second bilinear form

$$b_h(u, v) = \int_\Omega uv + \nabla u \cdot \nabla v \,\mathrm{d}\, x, \tag{17}$$

with corresponding matrix B, and operator $B_h : V_h \to V_h'$.

(i) Show that

$$B^{-1}A = \mathcal{I}_h^{-1} B_h^{-1} A_h \mathcal{I}_h. \tag{18}$$

[4 marks]

Solution: ***SEEN***

$$\begin{aligned}B^{-1}A &= ((\mathcal{I}_h')^{-1} B_h \mathcal{I}_h)^{-1} (\mathcal{I}_h')^{-1} A_h \mathcal{I}_h,\\ &= (\mathcal{I}_h)^{-1} B_h^{-1} \mathcal{I}_h' (\mathcal{I}_h')^{-1} A_h \mathcal{I}_h,\\ &= (\mathcal{I}_h)^{-1} B_h^{-1} A_h \mathcal{I}_h,\end{aligned}$$

as required.

(ii) Explain why $B_h^{-1}A_h$ has the same eigenvalues as $B^{-1}A$.

[3 marks]

Solution: ***SEEN***

$B^{-1}A$ *is similar to* $B_h^{-1}A_h$ *and so they have the same eigenvalues.*