1. This question is about the equation

$$-\nabla^2 u = f \text{ on } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a polygonal domain with boundary $\partial\Omega$.

(a) Let V be a continuous Lagrange finite element space defined on a triangulation of Ω. Describe how the finite element discretisation of (1) using V results in a matrix-vector equation

$$A\boldsymbol{u} = \boldsymbol{b}. \tag{2}$$

[10 marks]

(b) (i) Show that the matrix A satisfies

$$A\mathbf{1} = \mathbf{0}, \tag{3}$$

where $\mathbf{1}$ is the vector with all entries equal to 1, and $\mathbf{0}$ is the zero vector.

[2 marks]

(ii) Explain why this means that A is not invertible.

[1 marks]

(c) (i) Describe how to add an extra condition to Equation 1, and correspondingly to your finite element formulation, so that this issue is removed.

[2 marks]

(ii) Using the "mean estimate",

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)},$$

where $u \in V$ and $\bar{u}$ is the mean value of u, explain why Equation (3) cannot hold after modification.

[5 marks]

2. (a) Consider the finite element (K, P, N), where

- $*$ K is a triangle with vertices (z_1, z_2, z_3).
- $*$ P is the space of polynomials of degree 1 or less,
- $*$ $N = (N_1, N_2, N_3)$, where $N_i(p) = p(z_i)$, $i = 1, 2, 3$.

Show that N determines P.

[10 marks]

(a) Consider the finite element (K', Q, N'), where

- $*$ K' is a square with vertices (z_1, z_2, z_3, z_4) (enumerated clockwise around the square, starting at the bottom left).
- $*$ $Q = \text{Span}\{P, xy\}$, where P is the space of polynomials of degree 1 or less.
- $*$ $N' = (N_1, N_2, N_3, N_4)$, where $N_i(p) = p(z_i)$, $i = 1, 2, 3, 4$.

Show that N' determines Q.

[10 marks]

3. Consider the interval $[a, b]$, with points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$. Let $\mathcal{T}$ be a subdivision (i.e. a 1D mesh) of the interval $[a, b]$ into subintervals $I_k = [x_k, x_{k+1}]$, $k = 0, \ldots, N-1$. Consider the following three elements.

1. (K, P, N) where $K = I_k$, P are polynomials of degree ≤ 3, and $N = (N_1, N_2, N_3, N_4)$ with $N_1[u] = u(x_k)$, $N_2[u] = u(x_{k+1})$, $N_3[u] = \int_{x_k}^{x_{k+1}} u \,\mathrm{d}\, x$, $N_4[u] = u'((x_{k+1} + x_k)/2)$.
2. (K, P, N) where $K = I_k$, P are polynomials of degree ≤ 3, and $N = (N_1, N_2, N_3, N_4)$ with $N_1[u] = u(x_k)$, $N_2[u] = u(x_{k+1})$, $N_3[u] = u'(x_k)$, $N_4[u] = u'(x_{k+1})$.
3. (K, P, N) where $K = I_k$, P are polynomials of degree ≤ 3, and $N = (N_1, N_2, N_3, N_4)$ with $N_1[u] = u((x_{k+1} + x_k)/2)$, $N_2[u] = u'((x_{k+1} + x_k)/2)$, $N_3[u] = u''((x_{k+1} + x_k)/2)$, $N_4[u] = u'''((x_{k+1} + x_k)/2)$.

(a) Which of the three elements above are suitable for the following variational problem? Find $u \in H^1([a, b])$ such that

$$\int_a^b uv + u'v' \,\mathrm{d}\, x = \int_a^b fv \,\mathrm{d}\, x, \quad \forall v \in H^1([a, b]).$$

Justify your answer.

[10 marks]

(b) Which of the three elements above are suitable for the following variational problem? Find $u \in H^2([a, b])$ such that

$$\int_a^b uv + u'v' + u''v'' \,\mathrm{d}\, x = \int_a^b fv \,\mathrm{d}\, x, \quad \forall v \in H^2([a, b]).$$

Justify your answer.

[10 marks]

4. (a) For $f \in L^2(\Omega)$, where Ω is some convex polygonal domain, the L^2 projection of f into a degree k Lagrange finite element space V is the function $u \in V$ such that

$$\int_\Omega uv \,\mathrm{d}\,x = \int_\Omega vf \,\mathrm{d}\,x, \quad \forall v \in V.$$

Show that u exists and is unique from this definition, with

$$\|u\|_{L^2} \leq \|f\|_{L^2}.$$

[5 marks]

(b) Show that the L^2 projection is mean-preserving, i.e.

$$\int_\Omega u \,\mathrm{d}\,x = \int_\Omega f \,\mathrm{d}\,x.$$

[5 marks]

(c) Show that the L^2 projection u into V of f is the minimiser over $v \in V$ of the functional

$$J[v] = \int_\Omega (v - f)^2 \,\mathrm{d}\,x.$$

[5 marks]

(d) Hence, show that

$$\|u - f\|_{L^2(\Omega)} < Ch|f|_{H^1(\Omega)},$$

where h is the maximum triangle diameter in the triangulation used to construct V.

[5 marks]

(Mastery). We quote the following result from lectures. Let K_1 be a triangle with diameter 1, containing a ball B. There exists a constant C such that for $0 \leq |\beta| \leq k+1$ and all $f \in H^{k+1}(\Omega)$,

$$\|D^\beta(f - Q_{k,B}f)\|_{L^2(K_1)} \leq C\|\nabla^{k+1}f\|_{L^2(K_1)}, \quad (4)$$

where $Q_{k,B}$ is the degree-k ball-averaged Taylor polynomial of f.

(a) Let $\mathcal{I}_{K_1}$ be the nodal interpolation operator on K_1 for the Lagrange finite element of degree k. Using the following stability estimate

$$\|\mathcal{I}_K u\|_{H^k(K_1)} \leq C\|u\|_{H^k(K_1)},$$

when $k > 1$, together with the estimate in Equation (4), show that when $i \leq k$, we have

$$|\mathcal{I}_{K_1}u - u|_{H^i(K_1)} \leq C_1|u|_{H^{k+1}(K_1)}.$$

[5 marks]

(b) Let K be a triangle with diameter d. When $k > 1$ and $i \leq k$, show that

$$|\mathcal{I}_K u - u|_{H^i(K)} \leq d^{k+1-i}C_1|u|_{H^{k+1}(K)},$$

where C_1 is a constant that depends on the shape of K but not the size.

[5 marks]

(c) Let $\mathcal{T}$ be a triangulation such that the minimum aspect ratio r of the triangles K_i satisfies $r > 0$. Let V be the degree k Lagrange finite element space. Let $u \in H^{k+1}(\Omega)$. Let h be the maximum over all of the triangle diameters, assuming that with $0 \leq h < 1$. Show that for $i \leq k$ and $i < 2$, the global interpolation operator satisfies

$$\|\mathcal{I}_h u - u\|_{H^i(\Omega)} \leq Ch^{k+1-i}|u|_{H^{k+1}(\Omega)}. \quad (5)$$

[5 marks]

(d) Why does this estimate not hold for $i \geq 2$?

[5 marks]