1. This question is about the equation

\[-\nabla^2 u = f \quad \text{on } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega,\]

where \( \Omega \) is a polygonal domain with boundary \( \partial \Omega \).

(a) Let \( V \) be a continuous Lagrange finite element space defined on a triangulation of \( \Omega \). Describe how the finite element discretisation of (1) using \( V \) results in a matrix-vector equation

\[ Au = b. \]  

(2) [10 marks]

(b) (i) Show that the matrix \( A \) satisfies

\[ A1 = 0, \]

where \( 1 \) is the vector with all entries equal to 1, and \( 0 \) is the zero vector.

(3) [2 marks]

(ii) Explain why this means that \( A \) is not invertible. [1 marks]

(c) (i) Describe how to add an extra condition to Equation 1, and correspondingly to your finite element formulation, so that this issue is removed. [2 marks]

(ii) Using the “mean estimate”,

\[ \| u - \bar{u} \|_{L^2(\Omega)} \leq C | u |_{H^1(\Omega)}, \]

where \( u \in V \) and \( \bar{u} \) is the mean value of \( u \), explain why Equation (3) cannot hold after modification. [5 marks]
2. (a) Consider the finite element \( (K, P, N) \), where
   
   * \( K \) is a triangle with vertices \( (z_1, z_2, z_3) \).
   * \( P \) is the space of polynomials of degree 1 or less,
   * \( N = (N_1, N_2, N_3) \), where \( N_i(p) = p(z_i), i = 1, 2, 3. \)
   
   Show that \( N \) determines \( P \).

   [10 marks]

(a) Consider the finite element \( (K', Q, N') \), where
   
   * \( K' \) is a square with vertices \( (z_1, z_2, z_3, z_4) \) (enumerated clockwise around the square, starting at the bottom left).
   * \( Q = \text{Span}\{P, xy\} \), where \( P \) is the space of polynomials of degree 1 or less.
   * \( N' = (N_1, N_2, N_3, N_4) \), where \( N_i(p) = p(z_i), i = 1, 2, 3, 4. \)
   
   Show that \( N' \) determines \( Q \).

   [10 marks]
Consider the interval \([a, b]\), with points \(a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\). Let \(\mathcal{T}\) be a subdivision (i.e. a 1D mesh) of the interval \([a, b]\) into subintervals \(I_k = [x_k, x_{k+1}], k = 0, \ldots, N - 1\).

Consider the following three elements.

1. \((K, P, N)\) where \(K = I_k, P\) are polynomials of degree \(\leq 3\), and \(N = (N_1, N_2, N_3, N_4)\) with
\[N_1[u] = u(x_k), N_2[u] = u(x_{k+1}), N_3[u] = \int_{x_k}^{x_{k+1}} u \, dx, N_4[u] = u'(x_{k+1} + x_k)/2.\]

2. \((K, P, N)\) where \(K = I_k, P\) are polynomials of degree \(\leq 3\), and \(N = (N_1, N_2, N_3, N_4)\) with
\[N_1[u] = u(x_k), N_2[u] = u(x_{k+1}), N_3[u] = u'(x_k), N_4[u] = u'(x_{k+1}).\]

3. \((K, P, N)\) where \(K = I_k, P\) are polynomials of degree \(\leq 3\), and \(N = (N_1, N_2, N_3, N_4)\) with
\[N_1[u] = u((x_{k+1} + x_k)/2), N_2[u] = u'(x_{k+1} + x_k)/2, N_3[u] = u''((x_{k+1} + x_k)/2), N_4[u] = u'''((x_{k+1} + x_k)/2).\]

(a) Which of the three elements above are suitable for the following variational problem?
Find \(u \in H^1([a, b])\) such that
\[
\int_a^b uv + u'v' \, dx = \int_a^b f v \, dx, \quad \forall v \in H^1([a, b]).
\]
Justify your answer.

[10 marks]

(b) Which of the three elements above are suitable for the following variational problem?
Find \(u \in H^2([a, b])\) such that
\[
\int_a^b uv + u'v' + u''v'' \, dx = \int_a^b f v \, dx, \quad \forall v \in H^2([a, b]).
\]
Justify your answer.

[10 marks]
4. (a) For $f \in L^2(\Omega)$, where $\Omega$ is some convex polygonal domain, the $L^2$ projection of $f$ into a degree $k$ Lagrange finite element space $V$ is the function $u \in V$ such that

$$\int_\Omega uv \, dx = \int_\Omega vf \, dx, \quad \forall v \in V.$$ 

Show that $u$ exists and is unique from this definition, with

$$\|u\|_{L^2} \leq \|f\|_{L^2}.$$ 

[5 marks]

(b) Show that the $L^2$ projection is mean-preserving, i.e.

$$\int_\Omega u \, dx = \int_\Omega f \, dx.$$ 

[5 marks]

(c) Show that the $L^2$ projection $u$ into $V$ of $f$ is the minimiser over $v \in V$ of the functional

$$J[v] = \int_\Omega (v - f)^2 \, dx.$$ 

[5 marks]

(d) Hence, show that

$$\|u - f\|_{L^2(\Omega)} \leq Ch \|f\|_{H^1(\Omega)},$$

where $h$ is the maximum triangle diameter in the triangulation used to construct $V$.

[5 marks]
Mastery). We quote the following result from lectures. Let $K_1$ be a triangle with diameter 1, containing a ball $B$. There exists a constant $C$ such that for $0 \leq |\beta| \leq k + 1$ and all $f \in H^{k+1}(\Omega)$,

$$
\|D^\beta(f - Q_{k,B}f)\|_{L^2(K_1)} \leq C\|\nabla^{k+1}f\|_{L^2(K_1)},
$$

(4)

where $Q_{k,B}$ is the degree-$k$ ball-averaged Taylor polynomial of $f$.

(a) Let $I_{K_1}$ be the nodal interpolation operator on $K_1$ for the Lagrange finite element of degree $k$. Using the following stability estimate

$$
\|I_{K_1}u\|_{H^k(K_1)} \leq C\|u\|_{H^k(K_1)},
$$

when $k > 1$, together with the estimate in Equation (4), show that when $i \leq k$, we have

$$
|I_{K_1}u - u|_{H^i(K_1)} \leq C_1|u|_{H^{k+1}(K_1)}.
$$

[5 marks]

(b) Let $K$ be a triangle with diameter $d$. When $k > 1$ and $i \leq k$, show that

$$
|I_Ku - u|_{H^i(K)} \leq d^{k+1-i}C_1|u|_{H^{k+1}(K)},
$$

where $C_1$ is a constant that depends on the shape of $K$ but not the size.

[5 marks]

(c) Let $T$ be a triangulation such that the minimum aspect ratio $r$ of the triangles $K_i$ satisfies $r > 0$. Let $V$ be the degree $k$ Lagrange finite element space. Let $u \in H^{k+1}(\Omega)$. Let $h$ be the maximum over all of the triangle diameters, assuming that with $0 \leq h < 1$. Show that for $i \leq k$ and $i < 2$, the global interpolation operator satisfies

$$
\|I_hu - u\|_{H^i(\Omega)} \leq Ch^{k+1-i}|u|_{H^{k+1}(\Omega)}.
$$

(5)

[5 marks]

(d) Why does this estimate not hold for $i \geq 2$?

[5 marks]