1. This question is about the equation

$$-\nabla^2 u = f \text{ on } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a polygonal domain with boundary $\partial\Omega$.

(a) Let V be a continuous Lagrange finite element space defined on a triangulation of Ω. Describe how the finite element discretisation of (1) using V results in a matrix-vector equation

$$A\boldsymbol{u} = \boldsymbol{b}. \tag{2}$$

[10 marks]

Solution: ***SEEN***

First we develop the weak form by multiplying by a test function v, integrating by parts and removing the boundary integral due the Neumann boundary condition. The finite element discretisation is then: find $u \in V$ such that

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x - \int_\Omega v f \, \mathrm{d}\, x = 0, \quad \forall v \in V.$$

Let $\{\phi_i(x)\}_{i=1}^N$ be the nodal basis for V. Then expansion of v and u in the basis leads to

$$\sum_{i=1}^N v_i \left(\sum_{j=1}^N \int_\Omega \phi_i(x)\phi_j(x) \, \mathrm{d}\, x u_j - \int_\Omega \phi_i(x) f \, \mathrm{d}\, x \right) = 0,$$

but the v coefficients are arbitrary, so we have (2) with

$$A_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, \mathrm{d}\, x, \; x_i = u_i, b_i = \int_\Omega \phi_i f \, \mathrm{d}\, x.$$

(b) (i) Show that the matrix A satisfies

$$A\mathbf{1} = \mathbf{0}, \tag{3}$$

where $\mathbf{1}$ is the vector with all entries equal to 1, and $\mathbf{0}$ is the zero vector.

[2 marks]

Solution: ***UNSEEN***

$$(A\mathbf{1})_i = \int_\Omega \nabla\phi_i(x) \sum_{j=1}^N \nabla\phi_j(x).1 \, \mathrm{d}\, x = \int_\Omega \nabla\phi_i(x) \underbrace{\nabla(1)}_{=0} \mathrm{d}\, x = 0.$$

(ii) Explain why this means that A is not invertible.

[1 marks]

Solution: ***UNSEEN***

A is not invertible because it has a zero eigenvalue i.e. a nullspace.

(c) (i) Describe how to add an extra condition to Equation 1, and correspondingly to your finite element formulation, so that this issue is removed.

[2 marks]

Solution: ***SEEN***

We add an extra condition, that

$$\bar{u} = \int_\Omega u \, \mathrm{d}\, x = 0.$$

Then, we replace V with $\mathring{V}$ which is the subspace of V such that $\bar{u} = 0$ for all $u \in V$.

(ii) Using the "mean estimate",

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)},$$

where $u \in V$ and $\bar{u}$ is the mean value of u, explain why Equation (3) cannot hold after modification.

[5 marks]

Solution: ***UNSEEN***

Let A be the new matrix after reformulating with $\mathring{V}$ instead of V, under some basis. By contradiction: let $\boldsymbol{x}_0$ be a non-zero vector such that $A\boldsymbol{X}_0 = \mathbf{0}$. Then there exists a corresponding non-zero $u \in \mathring{V}$ such that

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x = 0, \quad \forall v \in V.$$

Taking $v = u$, we have

$$0 = \int_\Omega |\nabla u|^2 \, \mathrm{d}\, x := |u|^2_{H^1(\Omega)}.$$

Since u is non-zero, we have $\|u\|_{L^2} > 0$. Since $u \in \mathring{V}$, we have $\bar{u} = 0$. Hence we have

$$\|u - \bar{u}\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)} > 0.$$

This contradicts the mean estimate.

2. (a) Consider the finite element (K, P, N), where

- K is a triangle with vertices (z_1, z_2, z_3).
- P is the space of polynomials of degree 1 or less,
- $N = (N_1, N_2, N_3)$, where $N_i(p) = p(z_i)$, $i = 1, 2, 3$.

Show that N determines P.

[10 marks]

Solution: ***SEEN***

We make use of the result that if $p(x)$ is a degree k polynomial that vanishes on the line defined by $L(x) = 0$ and L is a non-degenerate affine polynomial, then $p(x) = L(x)q(x)$ where q is a polynomial of degree $k - 1$.

Let $p \in P$ such that $N_i(p) = 0$, $i = 1, 2, 3$. Let L_1 be a non-degenerate affine polynomial that vanishes on the line joining z_1 and z_2. Then the restriction of p to L_1 vanishes at 2 points and therefore is zero everywhere on L_1 by the fundamental theorem of algebra. Thus $p(x) = L_1(x)q(x)$ where q is a degree 0 polynomial, i.e. $p(x) = cL_1(x)$. We also have that $p(z_3) = 0$, and $L_1(x)$ does not vanish at z_3, so $c = 0$ i.e. $p = 0$ everywhere, hence N determines P.

(a) Consider the finite element (K', Q, N'), where

- K' is a square with vertices (z_1, z_2, z_3, z_4) (enumerated clockwise around the square, starting at the bottom left).
- $Q = \text{Span}\{P, xy\}$, where P is the space of polynomials of degree 1 or less.
- $N' = (N_1, N_2, N_3, N_4)$, where $N_i(p) = p(z_i)$, $i = 1, 2, 3, 4$.

Show that N' determines Q.

[10 marks]

Solution: ***SEEN SIMILAR***

We make use of the result that if $p(x)$ is a degree k polynomial that vanishes on the line defined by $L(x) = 0$ and L is a non-degenerate affine polynomial, then $p(x) = L(x)q(x)$ where q is a polynomial of degree $k - 1$.

Let $p \in P$ such that $N_i(p) = 0$, $i = 1, 2, 3, 4$. Let L_1 be a non-degenerate affine polynomial that vanishes on the line joining z_1 and z_2. Restricted to L_1, p is a degree 1 polynomial, since all elements of R are constant on L_1. Hence, $p(x) = L_1(x)q_1(x)$, where $q_1(x)$ has degree 1. Similarly, let L_2 be the non-degenerate affine polynomials vanishing on the line joining z_2 and z_3. The restriction of q_1 to that line vanishes at two points and is therefore equal to zero everywhere on that line, and hence $p(x) = cL_1(x)L_2(x)$. However, $p(z_4) = 0$, so $c = 0$ i.e. $p := 0$ i.e. Q determines N'.

3. Consider the interval $[a, b]$, with points $a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = b$. Let $\mathcal{T}$ be a subdivision (i.e. a 1D mesh) of the interval $[a, b]$ into subintervals $I_k = [x_k, x_{k+1}]$, $k = 0, \dots, N-1$. Consider the following three elements.

1. (K, P, N) where $K = I_k$, P are polynomials of degree ≤ 3, and $N = (N_1, N_2, N_3, N_4)$ with $N_1[u] = u(x_k)$, $N_2[u] = u(x_{k+1})$, $N_3[u] = \int_{x_k}^{x_{k+1}} u \, \mathrm{d}\, x$, $N_4[u] = u'((x_{k+1} + x_k)/2)$.
2. (K, P, N) where $K = I_k$, P are polynomials of degree ≤ 3, and $N = (N_1, N_2, N_3, N_4)$ with $N_1[u] = u(x_k)$, $N_2[u] = u(x_{k+1})$, $N_3[u] = u'(x_k)$, $N_4[u] = u'(x_{k+1})$.
3. (K, P, N) where $K = I_k$, P are polynomials of degree ≤ 3, and $N = (N_1, N_2, N_3, N_4)$ with $N_1[u] = u((x_{k+1} + x_k)/2)$, $N_2[u] = u'((x_{k+1} + x_k)/2)$, $N_3[u] = u''((x_{k+1} + x_k)/2)$, $N_4[u] = u'''((x_{k+1} + x_k)/2)$.

(a) Which of the three elements above are suitable for the following variational problem? Find $u \in H^1([a, b])$ such that

$$\int_a^b uv + u'v' \, \mathrm{d}\, x = \int_a^b fv \, \mathrm{d}\, x, \quad \forall v \in H^1([a, b]).$$

Justify your answer.

[10 marks]

Solution: ***SEEN SIMILAR***
This equation requires the finite element space to be in $H^1([a, b])$ which requires C^0 finite elements. Elements 1 and 2 can be used to make C^0 elements, because you can assign N_1 and N_2 to vertices a and b respectively in both cases, so vertex-assigned nodal variables determine the value of the function there. Element 3 cannot be used, as there is no C^1 geometric decomposition for it (all four nodal variables to determine values at a and b in both cases).

(b) Which of the three elements above are suitable for the following variational problem? Find $u \in H^2([a, b])$ such that

$$\int_a^b uv + u'v' + u''v'' \, \mathrm{d}\, x = \int_a^b fv \, \mathrm{d}\, x, \quad \forall v \in H^2([a, b]).$$

Justify your answer.

[10 marks]

Solution: ***SEEN SIMILAR***
This equation requires the finite element space to be in $H^2([a, b])$ which requires C^1 finite elements. Element 2 can be used to make C^1 elements, because you can assign N_1, N_3 and N_2, N_4 to vertices a and b respectively in both cases, so vertex-assigned nodal variables determine the value of the function and the derivative there.
Elements 1 and 3 cannot be used because the value of the derivatives at a and b require three nodal variables for each, so a C^1 geometric decomposition is not possible.

4. (a) For $f \in L^2(\Omega)$, where Ω is some convex polygonal domain, the L^2 projection of f into a degree k Lagrange finite element space V is the function $u \in V$ such that

$$\int_\Omega uv \,\mathrm{d}\,x = \int_\Omega vf \,\mathrm{d}\,x, \quad \forall v \in V.$$

Show that u exists and is unique from this definition, with

$$\|u\|_{L^2} \leq \|f\|_{L^2}.$$

[5 marks]

Solution: ***SEEN SIMILAR***

This variational problem has a bilinear form which is just the L^2 inner product. Hence it is trivially continuous and coercive with scaling constants equal to 1. From the Lax-Milgram theorem, the solution exists and is unique. Taking $v = u$, we have

$$\|u\|_{L^2}^2 = \langle u, f\rangle_{L^2} \leq \|u\|_{L^2}\|f\|_{L^2},$$

from Cauchy-Schwarz, and dividing both sides by $\|u\|_{L^2}$ gives the result.

(b) Show that the L^2 projection is mean-preserving, i.e.

$$\int_\Omega u \,\mathrm{d}\,x = \int_\Omega f \,\mathrm{d}\,x.$$

[5 marks]

Solution: ***UNSEEN***

Since V is a Lagrange finite element space of degree k, it contains the function $v = 1$, from which we obtain the result.

(c) Show that the L^2 projection u into V of f is the minimiser over $v \in V$ of the functional

$$J[v] = \int_\Omega (v - f)^2 \,\mathrm{d}\,x.$$

[5 marks]

Solution: ***UNSEEN***

Method 1: solve by computing variational derivative,

$$\delta J[v; \delta v] = 2\int_\Omega \delta v(v - f) \,\mathrm{d}\,x = 0, \quad \forall \delta v \in V,$$

which gives $v = u$.

Method 2: by contradiction. If u is not the minimiser, then there exists $v \in V$ with $J[v] \leq J[u]$. Then

$$\begin{aligned} J[v] = \int_\Omega (v - f)^2 \,\mathrm{d}\,x &= \int_\Omega ((v - u) + (u - f))^2 \,\mathrm{d}\,x, \\ &= \int_\Omega (v - u)^2 \,\mathrm{d}\,x + \underbrace{\int_\Omega 2(v - u)(u - f) \,\mathrm{d}\,x}_{=0 \text{ by defn of } u} + \int_\Omega (u - f)^2 \,\mathrm{d}\,x, \\ &= \|v - u\|_{L^2}^2 + J[u], \end{aligned}$$

and we conclude that $\|v - u\|_{L^2}^2 \leq 0$, a contradiction.

(d) Hence, show that

$$\|u - f\|_{L^2(\Omega)} < Ch|f|_{H^1(\Omega)},$$

where h is the maximum triangle diameter in the triangulation used to construct V.

[5 marks]

Solution: ***SEEN SIMILAR***

Since u minimises the functional J, we have

$$\begin{aligned}
\|u - f\|_{L^2(\Omega)} &= \sup_{\|v\|_{L^2(\Omega)}>0} \|v - f\|_{L^2(\Omega)}, \\
&\leq \|I_h f - f\|_{L^2(\Omega)}, \\
&\leq Ch|f|_{H^1(\Omega)},
\end{aligned}$$

where I_h is the nodal interpolation operator into V, and we used the standard approximation result for I_h.

(Mastery). We quote the following result from lectures. Let K_1 be a triangle with diameter 1, containing a ball B. There exists a constant C such that for $0 \leq |\beta| \leq k+1$ and all $f \in H^{k+1}(\Omega)$,

$$\|D^\beta(f - Q_{k,B}f)\|_{L^2(K_1)} \leq C\|\nabla^{k+1}f\|_{L^2(K_1)}, \tag{4}$$

where $Q_{k,B}$ is the degree-k ball-averaged Taylor polynomial of f.

(a) Let $\mathcal{I}_{K_1}$ be the nodal interpolation operator on K_1 for the Lagrange finite element of degree k. Using the following stability estimate

$$\|\mathcal{I}_K u\|_{H^k(K_1)} \leq C\|u\|_{H^k(K_1)},$$

when $k > 1$, together with the estimate in Equation (4), show that when $i \leq k$, we have

$$|\mathcal{I}_{K_1}u - u|_{H^i(K_1)} \leq C_1|u|_{H^{k+1}(K_1)}.$$

[5 marks]

Solution: ***SEEN***

$$\begin{aligned}
|\mathcal{I}_{K_1}u - u|^2_{H^i(K_1)} &\leq \|\mathcal{I}_{K_1}u - u\|^2_{H^{k+1}(K_1)} \\
&= \|\mathcal{I}_{K_1}u - Q_{k,B}u + Q_{k,B}u - u\|^2_{H^{k+1}(K_1)} \\
&\leq \|Q_{k,B}u - u\|^2_{H^{k+1}(K_1)} + \|\mathcal{I}(u - Q_{k,B}u)\|^2_{H^{k+1}(K_1)}, \\
&\leq \|Q_{k,B}u - u\|^2_{H^{k+1}(K_1)} + C^2\|Q_{k,B}u - u\|^2_{H^{k+1}(K_1)}, \\
&\leq (1 + C^2)|u|^2_{H^{k+1}(K_1)}.
\end{aligned}$$

(b) Let K be a triangle with diameter d. When $k > 1$ and $i \leq k$, show that

$$|\mathcal{I}_K u - u|_{H^i(K)} \leq d^{k+1-i}C_1|u|_{H^{k+1}(K)},$$

where C_1 is a constant that depends on the shape of K but not the size.

[5 marks]

Solution: ***SEEN***

Consider the change of variables $x \to \phi(x) = x/d$. This map takes K to K_1 with diameter 1. Then

$$\begin{aligned}
\int_K |D^\beta(I_K u - u)|^2 \,\mathrm{d}\,x &= d^{-2|\beta|+1}\int_{K_1} |D^\beta(I_{K_1}u\circ\phi - u\circ\phi)|^2 \,\mathrm{d}\,x, \\
&\leq C_1^2 d^{-2|\beta+1} \sum_{|\alpha|=k+1} \int_{K_1} |D^\alpha u\circ\phi|^2 \,\mathrm{d}\,x, \\
&\leq C_1^2 d^{-2|\beta+2(k+1)} \sum_{|\alpha|=k+1} \int_K |D^\alpha u|^2 \,\mathrm{d}\,x, \\
&= C_1^2 d^{2(-|\beta|+k+1)}|u|^2_{H^{k+1}(K)},
\end{aligned}$$

and taking the square root gives the result.

(c) Let $\mathcal{T}$ be a triangulation such that the minimum aspect ratio r of the triangles K_i satisfies $r > 0$. Let V be the degree k Lagrange finite element space. Let $u \in H^{k+1}(\Omega)$. Let h be the maximum over all of the triangle diameters, assuming that with $0 \leq h < 1$. Show that for $i \leq k$ and $i < 2$, the global interpolation operator satisfies

$$\|\mathcal{I}_h u - u\|_{H^i(\Omega)} \leq C h^{k+1-i} |u|_{H^{k+1}(\Omega)}. \tag{5}$$

[5 marks]

Solution: ***SEEN***

The Lagrange finite element space is C^0, so the first derivatives of $I_h u$ are defined in the finite element sense. Then we may write (for $i < 2$)

$$\begin{aligned}
\|\mathcal{I}_h u - u\|^2_{H^i(\Omega)} &= \sum_{K \in \mathcal{T}} \|\mathcal{I}_K u - u\|^2_{H^i(K)}, \\
&\leq \sum_{K \in \mathcal{T}} C_K d_K^{2(k+1-i)} |u|^2_{H^{k+1}(K)}, \\
&\leq C_{\max} h^{2(k+1-i)} \sum_{K \in \mathcal{T}} |u|^2_{H^{k+1}(K)}, \\
&= C_{\max} h^{2(k+1-i)} |u|^2_{H^{k+1}(\Omega)},
\end{aligned}$$

where the existence of the $C_{\max} = \max_K C_K < \infty$ is due to the lower bound in the aspect ratio.

(d) Why does this estimate not hold for $i \geq 2$?

[5 marks]

Solution: ***UNSEEN***

This is because the weak second derivatives of $I_h u$ are not in $L^2(\Omega)$, we only have $I_h u \in H^1(\Omega)$.