Imperial College London

Course:	M4MA47
Setter:	Colin Cotter
Checker:	David Ham
Editor:	Andrew Walton
External:	external
Date:	February 19, 2018

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2017

M4MA47

Finite elements: numerical analysis and implementation

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

Imperial College London

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite elements: numerical analysis and implementation

Date: ??

Time: ??

Time Allowed: ?? Hours

This paper has *??* Questions.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	$1/2$	1	$1\,1/2$	2	$2\,1/2$	3	$3\,1/2$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Let $(K, \mathcal{P}, \mathcal{N})$ be such that:

   1. K is a (non-degenerate) triangle,
   2. $\mathcal{P}$ is the space of polynomials of degree ≤ 1.
   3. $\mathcal{N}$ is the set of nodal variables with

$$N_1(u) = \int_K u \,\mathrm{d}\,x, \quad N_2(u) = \frac{\partial u}{\partial x}, \quad N_3(u) = \frac{\partial u}{\partial y}.$$

   Find the nodal basis for $\mathcal{P}$ corresponding to $\mathcal{P}$ by expanding in the monomial basis

$$\psi_1(x) = 1, \quad \psi_2(x) = x, \quad \psi_3(x) = y.$$

   (Hint: to make calculation easier, you may make use of the fact that matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

   form a group.)

   (b) Use this calculation to explain why $\mathcal{N}$ determines $\mathcal{P}$.

   (c) Comment on the relative sizes of the basis functions as the triangle area goes to zero, and suggest a rescaling of the nodal variables that remedies this.

2. We consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

1. K is a (non-degenerate) triangle,
2. $\mathcal{P}$ is the space $(P_1)^2$ of vector-valued polynomials (i.e. each vector component is in P_1).
3. Elements of $\mathcal{N}$ are dual functions that return the normal component of vector fields at the end of each edge (2 evaluations per edge, one at each end, and 3 edges, makes 6 dual functions in total).

The geometric decomposition of $(K, \mathcal{P}, \mathcal{N})$ is defined by associating each dual basis function with the edge where the normal component is evaluated.
We consider the finite element space V defined on a triangulation $\mathcal{T}$ of a polygonal domain Ω, constructed from the element above, so that dual basis evaluations agree for triangles on either side of each interior edge.

(a) Show that the weak divergence $\nabla_w \cdot u$ exists for $u \in V$, defined by

$$\int_\Omega \phi \nabla_w \cdot u \, \mathrm{d}\, x = -\int_\Omega \nabla \phi \cdot u \, \mathrm{d}\, x, \quad \forall \phi \in C_0^\infty(\Omega).$$

(b) Develop a variational formulation for the problem

$$u - c\nabla(\nabla \cdot u) = f, \text{ for } x \in \Omega, \quad u.n = 0 \text{ on } \partial\Omega,$$

using the finite element space V, for c a positive constant. Develop an inner product that gives coercivity and continuity for the corresponding bilinear form with respect to the corresponding normed space.

(c) Show that $u \in V$ does not have a weak curl $\nabla_w^\perp \cdot u$ in general, where

$$\int_\Omega \Phi \cdot \nabla_w^\perp \cdot u \, \mathrm{d}\, x = \int_\Omega \nabla^\perp \Phi \cdot u \, \mathrm{d}\, x, \quad \forall \Phi \in C_0^\infty(\Omega),$$

where $\nabla^\perp = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. [Hint: show this by counter-example. First choose a function $u \in V$ that you think does not have a weak curl. Then consider a suitable limit of smooth test functions that contradicts the above definition of a weak curl.]

3. We consider the following boundary value problem in one dimension.

$$-u'' + (2 + \sin(x))u = f(x), \quad u(0) = 0,\, u'(1) = 1.$$

(a) Construct a formulation of this problem describing a weak solution u in $H^1([0,1])$.

(b) Show that the corresponding bilinear form is continuous and coercive in $H^1([0,1])$, and compute the continuity and coercivity constants.

(c) What is the required property of f for a unique solution u to exist?

(d) Describe the piecewise linear C^0 finite element discretisation of this equation with mesh vertices $[x_0 = 0, x_1, x_2, \ldots, x_n, x_{n+1} = 1]$.

(e) Given an arbitrary basis of the finite element space V_h, show that the resulting matrix A is symmetric ($A^T = A$) and positive definite, i.e. $x^T A x > 0$ for all x with $\|x\| > 0$.

(f) Show that the numerical solution u_h satisfies $\|u - u_h\|_{H^1([0,1])} = \mathcal{O}(h)$ as $h \to 0$. [You may quote any properties of the interpolation operator $\mathcal{I}_h$ without proof, but must show the other steps.]

4. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

- K is a non-degenerate triangle,
- $\mathcal{P}$ is the space of polynomials on K of degree ≤ 1.
- $\mathcal{N} = (N_1, N_2, N_3)$, where

$$N_i(u) = \int_{f_i} u \,\mathrm{d}\, x,$$

where (f_1, f_2, f_3) are the edges of K, with f_1 joining vertices 1 and 2, f_2 joining vertices 2 and 3, and f_3 joining vertices 3 and 1.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$.

(b) We take a geometric decomposition such that N_i is associated with f_i, $i = 1, 2, 3$. What is the continuity of the corresponding finite element space V defined on a triangulation $\mathcal{T}$ of a polygonal domain Ω? Explain your answer.

Now consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

- K is a non-degenerate triangle,
- $\mathcal{P}$ is the space of polynomials on K of degree ≤ 2.
- $\mathcal{N} = (N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}, N_{3,1}, N_{3,2})$, where

$$N_{i,j}(u) = \int_{f_i} \phi_{i,j} u \,\mathrm{d}\, x,$$

where the edge test functions $\phi_{i,j}$ define a basis for linear functions restricted to f_i such that $\phi_{1,1} = 1$ on vertex 1 and 0 on vertex 2, *etc.*

(c) Show that $\mathcal{N}$ does not determine $\mathcal{P}$.