- 1. (a) Let $(K, \mathcal{P}, \mathcal{N})$ be such that:
 - 1. K is a (non-degenerate) triangle,
 - 2. \mathcal{P} is the space of polynomials of degree ≤ 1 .
 - 3. \mathcal{N} is the set of nodal variables with

$$N_1(u) = \int_K u \,\mathrm{d}\,x, \quad N_2(u) = \frac{\partial u}{\partial x}, \quad N_3(u) = \frac{\partial u}{\partial y}$$

Find the nodal basis for \mathcal{P} corresponding to \mathcal{N} by expanding in the monomial basis

$$\psi_1(x) = 1, \quad \psi_2(x) = x, \quad \psi_3(x) = y,$$

(Hint: to make calculation easier, you may make use of the fact that matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

form a group.)

- (b) Use this calculation to explain why \mathcal{N} determines \mathcal{P} .
- (c) Comment on the relative sizes of the basis functions as the triangle area goes to zero, and suggest a rescaling of the nodal variables that remedies this.

- 2. We consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where
 - 1. K is a (non-degenerate) triangle,
 - 2. \mathcal{P} is the space $(P_1)^2$ of vector-valued polynomials (i.e. each vector component is in P_1).
 - 3. Elements of \mathcal{N} are dual functions that return the normal component of vector fields at the end of each edge (2 evaluations per edge, one at each end, and 3 edges, makes 6 dual functions in total).

The geometric decomposition of $(K, \mathcal{P}, \mathcal{N})$ is defined by associating each dual basis function with the edge where the normal component is evaluated.

We consider the finite element space V defined on a triangulation \mathcal{T} of a polygonal domain Ω , constructed from the element above, so that dual basis evaluations agree for triangles on either side of each interior edge.

(a) Show that the weak divergence $\nabla_w \cdot u$ exists for $u \in V$, defined by

$$\int_{\Omega} \phi \nabla_w \cdot u \, \mathrm{d} \, x = - \int_{\Omega} \nabla \phi \cdot u \, \mathrm{d} \, x, \quad \forall \phi \in C_0^{\infty}(\Omega).$$

(b) Develop a variational formulation for the problem

$$u - c\nabla(\nabla \cdot u) = f$$
, for $x \in \Omega$, $u.n = 0$ on $\partial\Omega$,

using the finite element space V, for c a positive constant. Develop an inner product that gives coercivity and continuity for the corresponding bilinear form with respect to the corresponding normed space.

(c) Show that $u \in V$ does not have a weak curl $\nabla_w^{\perp} \cdot u$ in general, where

$$\int_{\Omega} \Phi \cdot \nabla_w^{\perp} \cdot u \, \mathrm{d} \, x = \int_{\Omega} \nabla^{\perp} \Phi \cdot u \, \mathrm{d} \, x, \quad \forall \Phi \in C_0^{\infty}(\Omega),$$

where $\nabla^{\perp} = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$. [Hint: show this by counter-example. First choose a function $u \in V$ that you think does not have a weak curl. Then consider a suitable limit of smooth test functions that contradicts the above definition of a weak curl.]

3. We consider the following boundary value problem in one dimension.

$$-u'' + (2 + \sin(x))u = f(x), \quad u(0) = 0, \ u'(1) = 1.$$

- (a) Construct a formulation of this problem describing a weak solution u in $H^1([0, 1])$.
- (b) Show that the corresponding bilinear form is continuous and coercive in $H^1([0,1])$, and compute the continuity and coercivity constants.
- (c) What is the required property of f for a unique solution u to exist?
- (d) Describe the piecewise linear C^0 finite element discretisation of this equation with mesh vertices $[x_0 = 0, x_1, x_2, \dots, x_n, x_{n+1} = 1]$.
- (e) Given an arbitrary basis of the finite element space V_h , show that the resulting matrix A is symmetric ($A^T = A$) and positive definite, i.e. $x^T A x > 0$ for all x with ||x|| > 0.
- (f) Show that the numerical solution u_h satisfies $||u u_h||_{H^1([0,1])} = \mathcal{O}(h)$ as $h \to 0$. [You may quote any properties of the interpolation operator \mathcal{I}_h without proof, but must show the other steps.]

- 4. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with
 - 1. K is a non-degenerate triangle,
 - 2. \mathcal{P} is the space of polynomials on K of degree ≤ 2 .
 - 3. $\mathcal{N} = (N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}, N_{3,1}, N_{3,2})$, where

$$N_{i,j}(u) = \int_{f_i} \phi_{i,j} u \,\mathrm{d}\, x,$$

where (f_1, f_2, f_3) are the edges of K, with f_1 joining vertices 1 and 2, f_2 joining vertices 2 and 3, and f_3 joining vertices 3 and 1. The edge test functions $\phi_{i,j}$ define a basis for linear functions restricted to f_i such that $\phi_{1,1} = 1$ on vertex 1 and 0 on vertex 2, *etc.*

- (a) Show that \mathcal{N} determines \mathcal{P} .
- (b) We take a geometric decomposition such that $N_{i,j}$ is associated with f_i , i = 1, 2, 3, j = 1, 2. What is the continuity of the corresponding finite element space V defined on a triangulation \mathcal{T} of a polygonal domain Ω ? Explain your answer.
- (c) We assume that $u \in H^k(\Omega)$ for some non-negative integer k. What is the minimum value of k such that the global interpolator $\mathcal{I}_h u$ is well-defined? Explain your answer.