Imperial College London

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2017

M4MA47

Finite elements: numerical analysis and implementation

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite elements: numerical analysis and implementation

Date: ??

Time: ??

Time Allowed: ?? Hours

This paper has ?? Questions.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	$1/2$	1	$1\,1/2$	2	$2\,1/2$	3	$3\,1/2$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Let $(K, \mathcal{P}, \mathcal{N})$ be such that:

1. K is a (non-degenerate) triangle,
2. $\mathcal{P}$ is the space of polynomials of degree ≤ 1.
3. $\mathcal{N}$ is the set of nodal variables with

$$N_1(u) = \int_K u \, \mathrm{d}\, x, \quad N_2(u) = \frac{\partial u}{\partial x}, \quad N_3(u) = \frac{\partial u}{\partial y}.$$

Find the nodal basis for $\mathcal{P}$ corresponding to $\mathcal{P}$ by expanding in the monomial basis

$$\psi_1(x) = 1, \quad \psi_2(x) = x, \quad \psi_3(x) = y.$$

(Hint: to make calculation easier, you may make use of the fact that matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

form a group.)

Solution: *We expand* $\phi_i(x) = \sum_{j=1}^{3} a_{ij}\psi_j(x)$. *Then,*

$$\delta_{ik} = N_k(\phi_i) = \sum_{j=1}^{3} a_{ij} N_k(\psi_j) = (AV)_{ik},$$

where A *is the matrix with coefficients* a_{ij}*, and* V *is the matrix with coefficients* $v_{ij} = N_j(\psi_i)$*, given by*

$$V = \begin{pmatrix} |K| & 0 & 0 \\ \bar{x}|K| & 1 & 0 \\ \bar{y}|K| & 0 & 1 \end{pmatrix}, \quad |K| = \int_K \mathrm{d}\, x, \; \bar{f} = \frac{\int_K f(x) \, \mathrm{d}\, x}{|K|}.$$

Using the hint, we write

$$V^{-1} = \begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix},$$

so

$$V^{-1}V = \begin{pmatrix} a|K| & 0 & 0 \\ |K|(b+\bar{x}) & 1 & 0 \\ |K|(c+\bar{y}) & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies V^{-1} = \begin{pmatrix} \frac{1}{|K|} & 0 & 0 \\ -\bar{x} & 1 & 0 \\ -\bar{y} & 0 & 1 \end{pmatrix},$$

hence we obtain the basis

$$\phi_1(x) = \frac{1}{|K|}, \; \phi_2(x) = x - \bar{x}, \; \phi_3(x) = y - \bar{y}.$$

(b) Use this calculation to explain why $\mathcal{N}$ determines $\mathcal{P}$.

Solution: *The functions $\{\phi_i\}_{i=1}^3$ satisfies the requirements of the dual basis and are written as an invertible transformation from the monomial basis, hence they are linearly independent and span $\mathcal{P}$.*

(c) Comment on the relative sizes of the basis functions as the triangle area goes to zero, and suggest a rescaling of the nodal variables that remedies this.

Solution: *As $|K|$ goes to zero, $\phi_1 \to 0$ whilst ϕ_2 and ϕ_3 stay finite. This can be fixed by replacing N_1 with $\tilde{N}_1 = \bar{u}$, in which case V becomes*

$$V = \begin{pmatrix} 1 & 0 & 0 \\ \bar{x} & 1 & 0 \\ \bar{y} & 0 & 1 \end{pmatrix},$$

so

$$\psi_1(x) = 1,\ \psi_2(x) = x - \bar{x},\ \psi_3(x) = y - \bar{y}.$$

2. We consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

   1. K is a (non-degenerate) triangle,
   2. $\mathcal{P}$ is the space $(P_1)^2$ of vector-valued polynomials (i.e. each vector component is in P_1).
   3. Elements of $\mathcal{N}$ are dual functions that return the normal component of vector fields at the end of each edge (2 evaluations per edge, one at each end, and 3 edges, makes 6 dual functions in total).

   The geometric decomposition of $(K, \mathcal{P}, \mathcal{N})$ is defined by associating each dual basis function with the edge where the normal component is evaluated.
   We consider the finite element space V defined on a triangulation $\mathcal{T}$ of a polygonal domain Ω, constructed from the element above, so that dual basis evaluations agree for triangles on either side of each interior edge.

   (a) Show that the weak divergence $\nabla_w \cdot u$ exists for $u \in V$, defined by

$$\int_\Omega \phi \nabla_w \cdot u \, \mathrm{d}\, x = - \int_\Omega \nabla \phi \cdot u \, \mathrm{d}\, x, \quad \forall \phi \in C_0^\infty(\Omega).$$

   Solution: *First we note that the normal components of $u \in V$ agree on both sides of each interior edge, since $u.n$ restricted to the edge is a linear scalar function. Since the values of $u.n$ on each side agree at two points, the difference $u^+.n^+ + u^- \cdot n^-$ is a linear function that vanishes at two points, and hence is zero.*
   We define

$$\nabla_w \cdot u|_K = \nabla \cdot u|_K,$$

   for all triangles $K \in \mathcal{T}$. Then

$$\begin{aligned} \int_\Omega \phi \nabla_w \cdot u \, \mathrm{d}\, x &= \sum_K \int_K \phi \nabla \cdot u \, \mathrm{d}\, x, \\ &= \sum_K \left(- \int_K \nabla \phi \cdot u \, \mathrm{d}\, x + \int_{\partial K} \phi n \cdot u \, \mathrm{d}\, S \right), \\ &= - \int_\Omega \nabla \phi \cdot u \, \mathrm{d}\, x, \end{aligned}$$

   as required, since the normal components agree.

   (b) Develop a variational formulation for the problem

$$u - c\nabla(\nabla \cdot u) = f, \text{ for } x \in \Omega, \quad u.n = 0 \text{ on } \partial\Omega,$$

   using the finite element space V, for c a positive constant. Develop an inner product that gives coercivity and continuity for the corresponding bilinear form with respect to the corresponding normed space.
   Solution: *Taking inner product with test function w, integration by parts and application of the boundary condition gives*

$$a(u, v) = F(v), \forall v \in V, \quad a(u, v) = \int_\Omega u.v + c\nabla \cdot u \nabla \cdot v \, \mathrm{d}\, x, \quad F(v) = \int_\Omega f \cdot v \, \mathrm{d}\, x.$$

$a(u, v)$ is symmetric, bilinear, positive-definite in L^2, thus we may use it as an inner product. a therefore has continuity and coercivity constants equal to 1 with respect to the corresponding norm.

(c) Show that $u \in V$ does not have a weak curl $\nabla_w^\perp \cdot u$ in general, where

$$\int_\Omega \Phi \cdot \nabla_w^\perp \cdot u \,\mathrm{d}\,x = \int_\Omega \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x, \quad \forall \Phi \in C_0^\infty(\Omega),$$

where $\nabla^\perp = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. [Hint: show this by counter-example. First choose a function $u \in V$ that you think does not have a weak curl. Then consider a suitable limit of smooth test functions that contradicts the above definition of a weak curl.]

Solution: *Let f_1 be an interior edge joining two triangles e_1 and e_2, and let f_2 be one of the other edges of e_2. Take $u.n = 1$ on f_2, and $u.n = 0$ on all other edges. Then, on the e_1 side of f_1, $u.t = 0$, where t is the unit tangent to f_1, oriented so that $n_1^\perp = t$, where n_1 is the unit normal pointing from e_1 to e_2. On the e_2 side, $u.t \neq 0$ (because otherwise $u.n = 0$ at the f_1 end of f_2). Define $g = u.t|_{e_2}$ on f_1. Then, pick a sequence of C_0^∞ test functions ϕ_k such that*

$$\phi_k|_{f_2} \to 1, \; \phi_k|_{e_1} \to 0 \, \phi_k|_{e_2} \to 0,$$

and $\phi_k \to 0$ in all triangles other than e_1, e_2. We have

$$\begin{aligned}\int_\Omega \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x &= \sum_K \int_K \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x, \\ &= \sum_K \left(- \int_K \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x + \int_{\partial K} \Phi n^\perp \cdot u \,\mathrm{d}\,S \right), \\ &\to \int_f g \Phi \,\mathrm{d}\,S \neq 0.\end{aligned}$$

On the other hand, if $\nabla_w^\perp u$ exists, then

$$0 \neq \int_f g\Phi \,\mathrm{d}\,S = \int_\Omega \Phi \nabla_w^\perp \cdot u \,\mathrm{d}\,S = \sum_K \int_\Omega \Phi|_K \nabla_w^\perp \cdot u \,\mathrm{d}\,S \to 0,$$

which is a contradiction.

3. We consider the following boundary value problem in one dimension.

$$-u'' + (2 + \sin(x))u = f(x), \quad u(0) = 0,\ u'(1) = 1.$$

(a) Construct a formulation of this problem describing a weak solution u in $H^1([0,1])$.
Solution: *Multiplication by test function v satisfying $v(0) = 0$, and integration by parts, using the boundary condition gives*

$$\int_0^1 v'u' + (2 + \sin(x))vu \,\mathrm{d}\,x = \int_0^1 vf \,\mathrm{d}\,x + v(1), \quad \forall v \in H_0^1([0,1]),$$

where $H_0^1([0,1])$ is the subspace of $H^1([0,1])$ such that $v(0)$.

(b) Show that the corresponding bilinear form is continuous and coercive in $H^1([0,1])$, and compute the continuity and coercivity constants.
Solution: *Continuity:*

$$\begin{aligned} a(u,v) &\leq \|v'\|_{L^2(\Omega)}\|u'\|_{L^2(\Omega)} + 3\|v\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}, \\ &\leq 3\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}. \end{aligned}$$

Constant is 3.
Coercivity:

$$\begin{aligned} a(u,u) &= \int_0^1 (u')^2 + (2 + \sin(x))u^2 \,\mathrm{d}\,x, \\ &\geq \int_0^1 (u')^2 + u^2 \,\mathrm{d}\,x = \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

Constant is 1.

(c) What is the required property of f for a unique solution u to exist?
Solution: *The RHS functional is*

$$L[v] = \int_0^1 fv \,\mathrm{d}\,x + v(1).$$

We have

$$|L[v]| \leq \|f\|_{L^2([0,1])}\|v\|_{L^2([0,1])} + C\|v\|_{H^1([0,1])} \leq (\|f\|_{L^2([0,1])} + C)\|v\|_{H^1([0,1])},$$

where C is the trace constant for $H^1([0,1])$. Hence we need $f \in L^2([0,1])$ for L to be bounded/continuous and hence the conditions of Lax-Migram to be satisfied.

(d) Describe the piecewise linear C^0 finite element discretisation of this equation with mesh vertices $[x_0 = 0, x_1, x_2, \ldots, x_n, x_{n+1} = 1]$.
Solution: *Let V_h be the finite element space of continuous, piecewise linear functions defined on this mesh, with subspace $\mathring{V}_h$ containing only the functions that satisfy $v_h(0) = 0$. The Galerkin finite element discretisation seeks $u_h \in \mathring{V}_h$ such that*

$$a(u_h, v) = L[v], \quad \forall v \in \mathring{V}_h.$$

(e) Given an arbitrary basis of the finite element space V_h, show that the resulting matrix A is symmetric ($A^T = A$) and positive definite, i.e. $x^T Ax > 0$ for all x with $\|x\| > 0$.

Solution: *Given a basis* $\{\phi_i\}_{i=1}^n$, A *is defined by*

$$A_{ij} = \int_0^1 \phi_i'\phi_j' + (2 + \sin(x))\phi_i\phi_j \,\mathrm{d}\, x.$$

A *is symmetric, since exchanging* i *and* j *returns the same answer.*

A *is PD since if we take*

$$u = \sum_{i=1}^n x_i\phi_i,$$

then

$$\begin{aligned} x^T Ax = \sum_{ij} x_i A_{ij} x_j &= \int_0^1 \sum_i x_i\phi_i' \sum_j x_j\phi_j' + (2 + \sin(x)) \sum_i x_i\phi_i \sum_j x_j\phi_j \,\mathrm{d}\, x, \\ &= \int_0^1 (u')^2 + (2 + sin(x))u^2 \,\mathrm{d}\, x, \end{aligned}$$

which is positive due to the coercivity result.

(f) Show that the numerical solution u_h satisfies $\|u - u_h\|_{H^1([0,1])} = \mathcal{O}(h)$ as $h \to 0$. [You may quote any properties of the interpolation operator $\mathcal{I}_h$ without proof, but must show the other steps.]

Solution: *Since* $\mathring{V}_h \in H^1([0, 1])$, *we have*

$$a(v, u - u_h) = 0, \quad \forall v \in \mathring{V}_h.$$

Then, taking $v \in \mathring{V}_h$,

$$\begin{aligned} \|u - u_h\|^2_{H^1([0,1])} &\leq a(u - u_h, u - u_h), \\ &= \underbrace{a(u - u_h, v - u_h)}_{=0} + a(u - u_h, u - v), \\ &\leq 3\|u - u_h\|_{H^1}\|u - v\|_{H^1}. \end{aligned}$$

Dividing by $\|u - u_h\|_{H^1}$ *and taking* $v = \mathcal{I}_h u$ *gives*

$$\|u - u_h\|_{H^1} \leq 3\|u - \mathcal{I}_h u\| \leq 3h\|u\|_{H^1},$$

where we quoted the property of $\mathcal{I}_h$ *in the last inequality.*

4. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

- K is a non-degenerate triangle,
- $\mathcal{P}$ is the space of polynomials on K of degree ≤ 1.
- $\mathcal{N} = (N_1, N_2, N_3)$, where

$$N_i(u) = \int_{f_i} u \, \mathrm{d}\, x,$$

where (f_1, f_2, f_3) are the edges of K, with f_1 joining vertices 1 and 2, f_2 joining vertices 2 and 3, and f_3 joining vertices 3 and 1.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$.

Solution: *It suffices to show that if $u \in P$, then $N_i(u) = 0$ for all $i \implies u = 0$.*

So, we assume that $u \in P$ with $N_i(u) = 0$, looking to show that $u = 0$. $N_i(u) = 0$ means that the average of u over the edge f_i is zero. Since u is linear on f_i, this means that u vanishes at the midpoint of each edge. These edges can be joined by three lines L_1, L_2, L_3, and we iteratively conclude that u vanishes on L_1 and L_2, so that $u = cL_1(x)L_2(x)$, and u vanishing on the third vertex not intersected by L_1 means that $c = 0$ (following the usual argument for linear Lagrange elements on triangles), and hence $u = 0$ everywhere.

(b) We take a geometric decomposition such that N_i is associated with f_i, $i = 1, 2, 3$. What is the continuity of the corresponding finite element space V defined on a triangulation $\mathcal{T}$ of a polygonal domain Ω? Explain your answer.

Solution: *There exist discontinuous functions in V, so functions are not even in $C^0(\Omega)$. To see this, pick two neighbouring triangles e_1, e_2 with common face f_1, and another face f_2 in e_1. We pick $u = 1$ at the midpoint of f_2 and $u = 0$ on all other midpoints. This function is equal to zero on the e_1 side of f_1, and is a linear function going from 1 to 0 on the e_2 side, hence discontinuous.*

Now consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

- K is a non-degenerate triangle,
- $\mathcal{P}$ is the space of polynomials on K of degree ≤ 2.
- $\mathcal{N} = (N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}, N_{3,1}, N_{3,2})$, where

$$N_{i,j}(u) = \int_{f_i} \phi_{i,j} u \, \mathrm{d}\, x,$$

where the edge test functions $\phi_{i,j}$ define a basis for linear functions restricted to f_i such that $\phi_{1,1} = 1$ on vertex 1 and 0 on vertex 2, *etc.*

(c) Show that $\mathcal{N}$ does not determine $\mathcal{P}$.

Solution: *We show by counter example. We take the quadratic q function that is equal to $1/6$ on each vertex, and $-1/12$ at each edge midpoint (this defines a unique quadratic function since these are the nodal variables for the standard Lagrange quadratic element, which is unisolvent). Consider one of the edges f_i, and choose a coordinate s which is equal* *to 0 on one end of the edge, and 1 on the other end. On that edge, $q|_{f_i}(s) = s^2 - s + 1/6$. This function has mean zero, and is symmetric, which means that*

$$\int_{f_i} \phi(s) q|_{f_i}(s) \,\mathrm{d}\, s = 0$$

for any linear function $\phi(s)$, and hence $N_{i,j}(u) = 0$, $j = 1, 2$. This means that all the nodal variables vanish when applied to u, but u is not zero. Hence, $\mathcal{N}$ does not determine $\mathcal{P}$.