Imperial College London

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Msci and MSc EXAMINATIONS (MATHEMATICS)

XXXX 2016

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Finite Elements: numerical analysis and implementation

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This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M4A47/M5A47

Finite Elements: numerical analysis and implementation

Date: XXXday, XX XXXXX 2016 Time: XX.00 Xm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly incomplete answers.

Calculators may not be used.

1. (a) Let V be the function space defined on $[0,1]$ by

$$V = \left\{ u \in L_2 : \int_0^1 u^2 + (u')^2 \,\mathrm{d}\,x < \infty \right\}.$$

Consider the variational problem,

$$\text{Find } u \in V \text{ such that } \int_0^1 uv + u'v' \,\mathrm{d}\,x = \int_0^1 vf \,\mathrm{d}\,x, \quad \forall v \in V. \qquad (1)$$

Let $0 < x_1 < x_2 < \ldots < x_{n-1} < 1$ define a subdivision of the interval $[0,1]$. Let V_h be a finite dimensional subspace of V, consisting of all functions that are linear in each subinterval, and continuous between subintervals. Formulate the finite element approximation for Equation (1) using S, and show how it results in a matrix-vector system of the form

$$K\mathbf{u} = \mathbf{F}.$$

[You do not need to compute the entries of K and $\mathbf{F}$, just provide a general formula for how they are calculated] **[5 Marks]**

(b) For the finite element approximation to Equation (1) given above, show that

$$\sum_{ij} K_{ij} = 1.$$

[5 Marks]

(c) Obtain all nodal basis functions for the finite element $(K = [0,1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$N_1(f) = f(0),$$
$$N_2(f) = f(1),$$
$$N_3(f) = \int_0^1 f \,\mathrm{d}\,x.$$

[5 Marks]

(d) What is the global continuity of finite element spaces constructed from the finite element described in part (c) of this question? Explain your answer. **[5 Marks]**

2. (a) Consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

- K is a non-degenerate triangle.
- $\mathcal{P}$ is the space of polynomials of degree 3 or less.
- The elements of $\mathcal{N}$ are: point evaluation at each of the vertices, gradient evaluation (both components) at each of the vertices, and point evalation at the centre of K.

Show that $\mathcal{N}$ determines $\mathcal{P}$. **[5 Marks]**

(b) Let $\mathcal{T}$ be a triangulation of a closed domain Ω, and let $\mathcal{I}_\mathcal{T}$ be the global interpolant from $C^m(\Omega)$ to the finite element space constructed from the element defined above on each triangle.

What is the value of m? **[5 Marks]**

(c) Show that $\mathcal{I}_\mathcal{T}$ is a C^0 interpolant.**[5 Marks]**

(d) Show that $\mathcal{I}_\mathcal{T}$ is not a C^1 interpolant.**[5 Marks]**

3. (a) Let $\mathcal{T}$ be a triangulation of a polygonal domain Ω, and let V_h be the degree k Lagrange finite element space defined by:

- $u \in V_h$ is a degree k polynomial when restricted to each triangle $T \in \mathcal{T}$,
- $u \in C^0(\Omega)$.

Show that $u \in V_h$ has weak partial derivatives and provide a formula for calculating it. **[10 Marks]**

(b) Now consider a different finite element space U_h defined by:

- $u \in U_h$ is a linear polynomial when restricted to each triangle $T \in \mathcal{T}$, and
- in each triangle T, the elements of the dual basis $\mathcal{N}_T$ are point evaluations at the midpoints of the three edges in T.

Is $u \in U_h$ a continuous function? Explain your answer. **[5 Marks]**

(c) Show that $u \in U_h$ does not have weak partial derivatives in general. [Hint: Show this by counter-example. First, choose a function which you think does not have a weak derivative. Then, consider a suitable limit of smooth test functions that contradicts the definition of a weak derivative.] **[5 Marks]**

4. (a) Let $(H, (\cdot,\cdot))$ be a Hilbert space, with closed subspace V. Let $a(u,v)$ be a (possibly not symmetric) bilinear form on V, and $F(v)$ be a continuous linear form on V. Let $\alpha > 0$, $C > 0$ be constants such that

$$|a(u,v)| \leq C\|u\|_V\|v\|_V, \forall u, v \in V,$$

and

$$a(v,v) \geq \alpha\|v\|_V^2, \quad \forall v \in V.$$

Let $F(u)$ be a continuous linear form on V. Let $u \in V$ be the solution of the variational problem

$$\text{Find } u \in V \text{ such that } a(u,v) = F(v), \quad \forall v \in V.$$

Let V_h be a finite dimensional subspace of V, so that u_h solves the Ritz-Galerkin variational problem

$$\text{Find } u_h \in V_h \text{ such that } a(u_h,v) = F(v), \quad \forall v \in V_h.$$

Show that

$$\|u - u_h\|_V \leq \frac{C}{\alpha}\min_{v \in V_h}\|u - v\|_V.$$

[6 Marks]

(b) Formulate the following differential equation as a variational problem on $V = H^1_{[0,1]}$.

$$-u'' + u' + u = f, \text{ on } [0,1], \quad u(0) = u(1) = 0. \tag{2}$$

[2 Marks]

(c) Show that the bilinear form from this variational problem satisfies the assumptions of Part (a) of this question. **[4 Marks]**

(d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ into elements with maximum width h. Let u_h be the solution to the the Ritz-Galerkin approximation of Equation (2) using V_h. Assuming the following result,

$$\min_{v \in V_h}\|u - v\|_{H^1_{[0,1]}} \leq h|u|_{H^2_{[0,1]}},$$

for $\gamma > 0$, show that

$$\|u - u_h\|_{H^1_{[0,1]}} \leq Dh|u|_{H^2_{[0,1]}},$$

and provide a numerical value for D. **[4 Marks]**

(e) Consider the modified variational problem for Equation (2) with boundary conditions $u'(0) = \alpha$, $u'(1) = \beta$. Show that this variational problem satisfies the conditions for Part (a) of this question. **[4 Marks]**