Imperial College London

Course:	M4A47/M5A47
Setter:	Colin Cotter
Checker:	David Ham
Editor:	Editor
External:	External
Date:	February 24, 2017

Msci and MSc EXAMINATIONS (MATHEMATICS)

XXXX 2016

M4A47/M5A47

Finite Elements: numerical analysis and implementation

Setter's signature	Checker's signature	Editor's signature
....	....	....

Imperial College London

MSci and MSc EXAMINATIONS (MATHEMATICS)

XXXX 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M4A47/M5A47

Finite Elements: numerical analysis and implementation

Date: XXXday, XX XXXXX 2016 Time: XX.00 Xm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly incomplete answers.

Calculators may not be used.

1. (a) Let V be the function space defined on $[0,1]$ by

$$V = \left\{ u \in L_2 : \int_0^1 u^2 + (u')^2 \,\mathrm{d}\, x < \infty \right\}.$$

Consider the variational problem,

$$\text{Find } u \in V \text{ such that } \int_0^1 uv + u'v' \,\mathrm{d}\, x = \int_0^1 vf \,\mathrm{d}\, x, \quad \forall v \in V. \tag{1}$$

Let $0 < x_1 < x_2 < \ldots < x_{n-1} < 1$ define a subdivision of the interval $[0,1]$. Let V_h be a finite dimensional subspace of V, consisting of all functions that are linear in each subinterval, and continuous between subintervals. Formulate the finite element approximation for Equation (1) using S, and show how it results in a matrix-vector system of the form

$$K\mathbf{u} = \mathbf{F}.$$

[You do not need to compute the entries of K and $\mathbf{F}$, just provide a general formula for how they are calculated]

Solution: SEEN

Let $\{\phi_i\}_{i=1}^m$ be a basis for S. Expanding u and v in the basis with coefficients u_i, v_i respectively, Equation (1) becomes

$$\sum_i v_i \left(\sum_j \int_0^1 \phi_i \phi_j + \phi_i' \phi_j' \,\mathrm{d}\, x u_j - \int_0^1 \phi_i f \,\mathrm{d}\, x \right) = 0.$$

Since this must be true for all basis coefficients v_i, we have

$$\sum_j \underbrace{\int_0^1 \phi_i \phi_j + \phi_i' \phi_j' \,\mathrm{d}\, x}_{=K_{ij}} u_j = \underbrace{\int_0^1 \phi_i f \,\mathrm{d}\, x}_{=F_i},$$

which takes the required form. **[5 Marks]**

(b) For the finite element approximation to Equation (1) given above, show that

$$\sum_{ij} K_{ij} = 1.$$

[5 Marks]

Solution: NOT SEEN

Since the global nodal basis satisfies $\sum_{i=1}^m \phi_i = 1$, we therefore have $\sum_{i=1}^m \phi_i' = 0$. Hence,

$$\begin{aligned}\sum_{ij} K_{ij} &= \sum_{ij} \int_0^1 \phi_i \phi_j + \phi_i' \phi_j' \,\mathrm{d}\, x, \\ &= \int_0^1 (\sum_i \phi_i)(\sum_j \phi_j) + (\sum_i \phi_i')(\sum_j \phi_j') \,\mathrm{d}\, x = 1.\end{aligned}$$

(c) Obtain all nodal basis functions for the finite element $(K = [0,1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$N_1(f) = f(0),$$
$$N_2(f) = f(1),$$
$$N_3(f) = \int_0^1 f \,\mathrm{d}\, x.$$

[5 Marks]

Solution: SEEN SIMILAR

We write

$$\phi_1(x) = (ax+1)(1-x) = 1 + (a-1)x - ax^2.$$

so that $\phi_1(0) = 1$, $\phi_1(1) = 0$. *Then we require*

$$\int_0^1 \phi_1(x) \,\mathrm{d}\, x = a/6 + 1/2 = 0,$$

and hence $a = -3$, *and* $\phi_1 = (3x+1)(1-x)$. *By symmetry,* $\phi_2(x) = \phi_1(1-x) = (3x-2)x$.

We may write $\phi_3(x) = cx(1-x)$, *then*

$$1 = \int_0^1 \phi_3(x) \,\mathrm{d}\, x = c/6 \implies c = 6,$$

so $\phi_3 = 6x(1-x)$.

(d) What is the global continuity of finite element spaces constructed from the finite element described in part (c) of this question? Explain your answer. **[5 Marks]**

Solution: NOT SEEN

The functions are continuous since they agree at each interface between subdivisions. This is because all basis functions vanish at the interface except for the basis function corresponding to point evaluation at the interface, which agrees on both sides.

2. (a) Consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

- K is a non-degenerate triangle.
- $\mathcal{P}$ is the space of polynomials of degree 3 or less.
- The elements of $\mathcal{N}$ are: point evaluation at each of the vertices, gradient evaluation (both components) at each of the vertices, and point evalation at the centre of K.

Show that $\mathcal{N}$ determines $\mathcal{P}$. **[5 Marks]**

Solution: SEEN

It is sufficient to show that $N_i(p) = 0,\ i = 1, \ldots, 10 \implies p = 0 \quad \forall p \in \mathcal{P}$.
We use the fact that if a degree p *polynomial* $P(x)$ *vanishes on a line defined by* $L(x) = 0$ *for a non-degenerate linear polynomial* L*, then*

$$P(x) = L(x)Q(x),$$

for some degree $p-1$ *polynomial* $Q(x)$.
Let $L_1(x)$ *be the non-degenerate linear polynomial that vanishes on the line intersecting vertices* z_2 *and* z_3. $P(x)$ *restricted to that line is a polynomial that vanishes at the endpoints, plus the derivative along the line vanishes (since both components of the derivative vanish, so directional derivatives vanish). Since* P *is cubic when restricted to the line* L_1*, and has double roots at each vertex,* P *vanishes on* L_1 *and hence Hence,*

$$P(x) = L_1(x)Q(x),$$

where $Q(x)$ *is a quadratic polynomial.*
By a similar argument applied iteratively to the other two sides, we obtain

$$P(x) = cL_1(x)L_2(x)L_3(x),$$

where c *is a constant. Neither* $L_1(x)$, $L_2(x)$ *nor* $L_3(x)$ *vanish at the triangle midpoint, but* $P(x)$ *must vanish there, so* $c = 0$*, implying that* $P(x) \equiv 0$ *as required.*

(b) Let $\mathcal{T}$ be a triangulation of a closed domain Ω, and let $\mathcal{I}_{\mathcal{T}}$ be the global interpolant from $C^m(\Omega)$ to the finite element space constructed from the element defined above on each triangle.

What is the value of m? **[5 Marks]**

Solution: SEEN *The dual basis requires derivative evaluation and thus* $m = 1$.

(c) Show that $\mathcal{I}_{\mathcal{T}}$ is a C^0 interpolant.**[5 Marks]**

Solution: SEEN SIMILAR

It is sufficient to show that the stated continuity holds across each edge e *shared by triangles* T_1 *and* T_2*, with finite elements* $(T_1, P_3, \mathcal{N}_1)$ *and* $(T_1, P_3, \mathcal{N}_2)$ *of type described above. Since the nodes on each side of* e *are located on the edge vertices, the nodal values coincide. For any function* $p \in C^1(\Omega)$*, let* $p_1 = \mathcal{I}_{T_1}p$*, and* $p_2 = \mathcal{I}_{T_2}p$*. Let* $w = p_1|_e - p_2|_e$*. Then* w *is cubic polynomial with double roots at each edge vertex*

(d) Show that $\mathcal{I}_{\mathcal{T}}$ is not a C^1 interpolant. **[5 Marks]**

Solution: NOT SEEN

Counter-example: let u be equal to zero in T_1, but equal to the cubic "bubble" function in T_2 that vanishes on all edges of T_2. The bubble function vanishes at the vertices. Further, it vanishes along each edge of T_2, and so the directional derivative tangential to each edge vanishes. For non-degenerate triangles, this means that both components of the derivative vanish at each vertex. This means that the edge nodes agree between T_1 and T_2; however, the normal derivative is not zero at the centre of e on the T_2 side, so the function is not in C^1.

3. (a) Let $\mathcal{T}$ be a triangulation of a polygonal domain Ω, and let V_h be the degree k Lagrange finite element space defined by:

- $u \in V_h$ is a degree k polynomial when restricted to each triangle $T \in \mathcal{T}$,
- $u \in C^0(\Omega)$.

Show that $u \in V_h$ has weak partial derivatives and provide a formula for calculating it. **[10 Marks]**

Solution: SEEN

Let $u \in V_h$. We claim that the weak partial derivatives of u are characterised as members of $L^1_{loc}(\Omega)$ satisfying

$$D_i^w u|_T = \frac{\partial}{\partial x_i} u|_T, \quad \forall T \in \mathcal{T}.$$

To check this, let $\phi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}
\int_\Omega \phi D_i^w u \, \mathrm{d}\, x &= \sum_{T \in \mathcal{T}} \int_T \phi D_i^w u \, \mathrm{d}\, x, \\
&= \sum_{T \in \mathcal{T}} \int_T \phi \frac{\partial}{\partial x_i} u \, \mathrm{d}\, x, \\
&= \sum_{T \in \mathcal{T}} \left(- \int_T \frac{\partial}{\partial x_i} \phi u \, \mathrm{d}\, x + \int_{\partial T} n_i \phi u \, \mathrm{d}\, x \right), \\
&= - \sum_{T \in \mathcal{T}} \int_T \frac{\partial}{\partial x_i} \phi u \, \mathrm{d}\, x + \sum_{e \in \Gamma} \int_e n_i^+ \underbrace{(\phi^+ u^+ - \phi^- u^-)}_{=0} \mathrm{d}\, S + \int_{\partial\Omega} n_i \underbrace{\phi}_{=0} u \, \mathrm{d}\, S, \\
&= - \int_\Omega \frac{\partial}{\partial x_i} \phi u \, \mathrm{d}\, x,
\end{aligned}$$

as required. Here, ∂T is the boundary of triangle T, Γ is the set of interior edges of $\mathcal{T}$ with each edge arbitrarily allocated a $+$ and $-$ side, n_i is the ith component of the unit outward pointing normal to ∂T, n_i^+ is the normal to edge e pointing from the $+$ side into the $-$ side, $\phi^+ u^+$ and $\phi^- u^-$ are the values of ϕu on the $+$ and $-$ sides respectively, and $\partial\Omega$ is the boundary of Ω with unit outward pointing normal with components n_i.

(b) Now consider a different finite element space U_h defined by:

- $u \in U_h$ is a linear polynomial when restricted to each triangle $T \in \mathcal{T}$, and
- in each triangle T, the elements of the dual basis $\mathcal{N}_T$ are point evaluations at the midpoints of the three edges in T.

Is $u \in U_h$ a continuous function? Explain your answer. **[5 Marks]**

Solution: NOT SEEN

It is not in general. For two triangles T_1 and T_2 joined by an edge e, Consider u defined by:

- *$u = 0$ on T_1,*
- *On T_2 with edges e, f, g, $u = 0$ in the midpoints of e and f, and $u = 1$ on the midpoint of g. Then $u = 1$ on the vertex joining e and g, and $u = -1$ on the vertex*

(c) Show that $u \in U_h$ does not have weak partial derivatives in general. [Hint: Show this by counter-example. First, choose a function which you think does not have a weak derivative. Then, consider a suitable limit of smooth test functions that contradicts the definition of a weak derivative.] **[5 Marks]**

Solution: NOT SEEN

Consider for example the function u defined above, on the unit square Ω subdivided into two right angled triangles with e going from top left to bottom right, T_1 is the lower triangle and T_2 is the upper triangle. Let edge g be the top edge of the square.

Then in T_2, $u = 1 - 2y$.

For any $\phi \in C_0^\infty$, we have

$$\int_\Omega \frac{\partial}{\partial x}\phi u \, \mathrm{d}\, x = -\int_e \phi(1-2y)\frac{1}{\sqrt{2}} \, \mathrm{d}\, S.$$

If u has a weak x-derivative v, then it satisfies

$$\int_\Omega \phi v \, \mathrm{d}\, x = \int_e \phi(1-2y)\frac{1}{\sqrt{2}} \, \mathrm{d}\, S.$$

Now choose a sequence of functions $\phi_n \in C_0^\infty$, satisfying $0 \leq \phi_n \leq 1$, with $\phi(x, y) = (1 - 2y)$ for $(x, y) \in e \cap \{(x, y) : (x - 0.5)^2 + (y - 0.5)^2 < 0.5\}$, such that $\phi_n(x, y) \to 0$ as $n \to \infty$ for $(x, y) \neq e$. Then

$$\int \phi_n v \, \mathrm{d}\, x \to 0,$$

which leads to a contradiction.

4. (a) Let $(H, (\cdot, \cdot))$ be a Hilbert space, with closed subspace V. Let $a(u, v)$ be a (possibly not symmetric) bilinear form on V, and $F(v)$ be a continuous linear form on V. Let $\alpha > 0$, $C > 0$ be constants such that

$$|a(u, v)| \leq C\|u\|_V\|v\|_V, \forall u, v \in V,$$

and

$$a(v, v) \geq \alpha\|v\|_V^2, \quad \forall v \in V.$$

Let $F(u)$ be a continuous linear form on V. Let $u \in V$ be the solution of the variational problem

$$\text{Find } u \in V \text{ such that } a(u, v) = F(v), \quad \forall v \in V.$$

Let V_h be a finite dimensional subspace of V, so that u_h solves the Ritz-Galerkin variational problem

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v) = F(v), \quad \forall v \in V_h.$$

Show that

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V.$$

[6 Marks]

Solution: SEEN

From coercivity, for any $v \in V_h$ we have,

$$\begin{aligned} \alpha\|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h), \\ \textit{[bilinearity]} &= a(u - u_h, u - v) + a(u - u_h, v - u_h), \\ \textit{[Galerkin orthogonality]} &= a(u - u_h, u - v), \\ \textit{[continuity]} &\leq C\|u - u_h\|_V\|u - v\|_V. \end{aligned}$$

Dividing by α and minimising over all $v \in V_h$, we get the result.

(b) Formulate the following differential equation as a variational problem on $V = H^1_{[0,1]}$.

$$-u'' + u' + u = f, \text{ on } [0, 1], \quad u(0) = u(1) = 0. \qquad (2)$$

[2 Marks]

Solution: SEEN

Multiplying by $v \in V$, integrating by parts and dropping the boundary terms due to the boundary conditions, we get

$$\int_0^1 (u'v' + u'v + uv) \, \mathrm{d}\, x = \int_0^1 fv \, \mathrm{d}\, x,$$

which matches the pattern with

$$a(u, v) = \int_0^1 (u'v' + u'v + uv) \, \mathrm{d}\, x, \quad F(v) = \int_0^1 fv \, \mathrm{d}\, x$$

(c) Show that the bilinear form from this variational problem satisfies the assumptions of Part (a) of this question. **[4 Marks]**

Solution: SEEN

First check F is continuous:

$$\text{[Cauchy-Schwarz] } |F(v)| \leq \|f\|_{L^2}\|v\|_{L^2},$$
$$\text{[definition of } H^1 \text{ norm]} \leq \|f\|_{L^2}\|v\|_{H^1}.$$

SEEN SIMILAR

Now check continuity:

$$\text{[Triangle inequality] } |a(u,v)| \leq |(u,v)_{H^1}| + |\int_0^1 u'v \,\mathrm{d}\,x|$$
$$\text{[Cauchy-Schwarz] } \leq \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L^2}$$
$$\text{[definition of } H^1 \text{ norm]} \leq 2\|u\|_{H^1}\|v\|_{H^1},$$

so $C = 2$. **SEEN SIMILAR**

Now check coercivity:

$$a(v,v) = \int_0^1 (v')^2 + v'v + v^2 \,\mathrm{d}\,x,$$
$$\text{[completing the square]} = \int_0^1 \underbrace{(v'+v)^2}_{\geq 0} \,\mathrm{d}\,x + \frac{1}{2}\int_0^1 ((v')^2 + v^2)\,\mathrm{d}\,x,$$
$$\geq \frac{1}{2}\|v\|_{H^1}.$$

(d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ into elements with maximum width h. Let u_h be the solution to the the Ritz-Galerkin approximation of Equation (2) using V_h. Assuming the following result,

$$\min_{v \in V_h} \|u - v\|_{H^1_{[0,1]}} \leq h|u|_{H^2_{[0,1]}},$$

for $\gamma > 0$, show that

$$\|u - u_h\|_{H^1_{[0,1]}} \leq Dh|u|_{H^2_{[0,1]}},$$

and provide a numerical value for D. **[4 Marks]**

Solution: SEEN SIMILAR

Using the result of Part (a), plus $C = 2$ and $\alpha = 1/2$ from Part (b), we have

$$\|u - u_h\|_V \leq \frac{2}{1/2}\min_{v \in V_h}\|u - v\|_V,$$
$$\text{[Error estimate] } \leq 4h|u|_{H^1_{[0,1]}},$$

hence $D = 4$.

(e) Consider the modified variational problem for Equation (2) with boundary conditions $u'(0) = \alpha$, $u'(1) = \beta$. Show that this variational problem satisfies the conditions for Part (a) of this question. **[4 Marks]**

Solution: NOT SEEN

After integration by parts we get

$$\int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x = \int_0^1 fv\,\mathrm{d}\,x - \beta v(1) + \alpha v(0).$$

The only change to the variational problem is that now

$$F(v) = (f, v) - \beta v(1) + \alpha v(0).$$

We have the trace estimate

$$|v(0)| + |v(1)| \le \delta \|v\|_{H^1},$$

hence

$$|F(v)| \le (\|f\|_{L^2} + \max(\beta, \alpha))\,\|v\|_{H^1},$$

and hence F is still continuous.