Imperial College London

Course:	M5MA47
Setter:	Colin Cotter
Checker:	David Ham
Editor:	Editor
External:	External
Date:	March 7, 2016

MSc EXAMINATIONS (MATHEMATICS)

XXXX 2015

M5MA47

Finite Elements: numerical analysis and implementation

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

Imperial College London

MSc EXAMINATIONS (MATHEMATICS)

XXXX 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M5MA47

Finite Elements: numerical analysis and implementation

Date: XXXday, XX XXXXX 2015 Time: XX.00 Xm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly incomplete answers.

Calculators may not be used.

1\. 1. Provide a variational formulation for the following equation for u.

$$u'' - u = -f, \quad u(0) = 0, \, u'(1) = 1.$$

Solution: SEEN SIMILAR

Multiply by test function v that satisfies $v(0) = 0$, integrate by parts to get

$$\int_0^1 v'u' + vu \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x + [vu']_0^1.$$

[2 Marks]

Then, applying the boundary condition we get

$$\int_0^1 v'u' + vu \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x + v(1).$$

[1 Marks]

Defining

$$a(u, v) = \int_0^1 v'u' + vu \, \mathrm{d}x, \quad F(v) = \int_0^1 fx \, \mathrm{d}x + v(1),$$

and

$$V = \{u : a(u, u) < \infty : u(0) = 0\},$$

the problem becomes to find $u \in V$ such that

$$a(u, v) = F(v), \quad \forall v \in V.$$

[2 Marks]

2. A quadrature rule on a reference element K has degree of precision m if it produces the exact answer for all polynomials of degree m or less. What is the minimum degree of precision required to exactly assemble the matrix for the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx,$$

on a mesh of elements that are affine equivalent to the finite element $(K, \mathcal{P}_2, \mathcal{N})$, where $\mathcal{P}_2$ are the polynomials of degree 2 or less, and $\mathcal{N}$ is the dual basis corresponding to evaluation at triangle vertices and midpoints.

Solution: SEEN SIMILAR

The transformation to a reference element results in integrals of the form

$$\int_K |\det J| \nabla \bar{u} \cdot \nabla \bar{v} \, \mathrm{d}x,$$

where $\bar{u}$ and $\bar{v}$ are the pullbacks of u and v to K, respectively. Since the transformation is affine, $|\det J|$ is a constant. Further, since u and v are polynomials of degree 2 or less, so are $\bar{u}$ and $\bar{v}$ (by affine-equivalence). Hence ∇u and ∇v are polynomials of degree 1 or less, and hence we need to integrate a polynomial of degree 2 or less, i.e. *we need a quadrature rule of degree 2.* **[5 Marks]**

3. Obtain the nodal basis function $\phi_1(x)$ for the finite element $(K = [0,1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$N_1(f) = f'(0),$$
$$N_2(f) = f'(1),$$
$$N_3(f) = \int_0^1 f dx.$$

Solution: SEEN SIMILAR

We write

$$\phi_1(x) = ax^2 + bx + c,$$

so

$$\phi_1'(x) = 2ax + b.$$

The first two dual basis functions imply that $\phi_1'(0) = 1$, $\phi_1'(1) = 0$, *so we obtain* $b = 1$, $a = -1/2$. *Then*

$$\int_0^1 \phi_1(x)\,\mathrm{d}\,x = \int_0^1 \frac{-x^2}{2} + x + c\,\mathrm{d}\,x = -\frac{1}{6} + \frac{1}{2} + c = 0,$$

so $c = -1/3$, *hence*

$$\phi_1(x) = -\frac{x^2}{2} + x - \frac{1}{3}.$$

[5 Marks]

4. What is the global continuity of finite element spaces constructed from the finite element described in part (3) of this question? Explain your answer.

Solution: NOT SEEN

The finite element functions are not even C^0. *This is because the basis function* $\phi_3(x) = 1$, *and so it contributes to the value of finite element functions at* $x = 0$ *and* $x = 1$ *in the reference element.* **[5 Marks]**

2. 1. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

- K is a non-degenerate triangle.
- $\mathcal{P}$ is the space of polynomials of degree 2 or less.
- $\mathcal{N} = (N_1, N_2, N_3, N_4, N_5, N_6)$ with

$$N_i(v) = v(z_i),\ i = 1, 2, 3,$$
$$N_4(v) = v\left(\frac{z_1 + z_2}{2}\right),$$
$$N_5(v) = v\left(\frac{z_1 + z_3}{2}\right),$$
$$N_6(v) = v\left(\frac{z_2 + z_3}{2}\right),$$

where z_1, z_2 and z_3 are the vertices of K.

Show that $\mathcal{N}$ determines $\mathcal{P}$.

Solution: SEEN

It is sufficient to show that $N_i(p) = 0\, i = 1, \ldots, 6 \implies p = 0 \quad \forall p \in \mathcal{P}$. **[1 Marks]**

We use the fact that if a degree p *polynomial* $P(x)$ *vanishes on a line defined by* $L(x) = 0$ *for a non-degenerate linear polynomial* L*, then*

$$P(x) = L(x)Q(x),$$

for some degree $p - 1$ *polynomial* $Q(x)$. **[3 Marks]**

Let $L_1(x)$ *be the non-degenerate linear polynomial that vanishes on the line intersecting* z_2 *and* z_3. $P(x)$ *restricted to that line is a quadratic polynomial that vanishes at three points, hence is zero by fundamental theorem of algebra. Hence,*

$$P(x) = L_1(x)Q(x),$$

where $Q(x)$ *is a linear polynomial.* **[2 Marks]**

By a similar argument we get

$$Q(x) = cL_2(x),$$

where c *is a constant and* $L_2(x)$ *is a non-degenerate linear polynomial that vanishes on the line intersecting* z_1 *and* z_3. **[2 Marks]**

Hence,

$$P(x) = cL_1(x)L_2(x).$$

Neither $L_1(x)$ *nor* $L_2(x)$ *vanish at* $x = (z_1 + z_2)/2$*, but* $P(x)$ *must vanish there, so* $c = 0$*, implying that* $P(x) \equiv 0$ *as required.* **[2 Marks]**

2. Now consider the finite element $(K, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ where

- K is a non-degenerate triangle (with boundary ∂K).
- $\hat{\mathcal{P}}$ is the space spanned by polynomials of degree 2 or less, plus the cubic "bubble" function $B(x)$ satisfying $B(x) = 0$ for all $x \in \partial K$, and $\int_K B(x) \,\mathrm{d}\, x = 1$.

– $\hat{\mathcal{N}} = (N_1, N_2, N_3, N_4, N_5, N_6, N_7)$ with N_i as above for $i = 1, \ldots, 6$, and

$$N_7(v) = v\left(\frac{z_1 + z_2 + z_3}{3}\right).$$

Show that $\hat{N}$ determines $\hat{P}$.

Solution: NOT SEEN

We assume that we have a polynomial $P \in \hat{P}$ that vanishes under the nodal basis $\mathcal{N}$. Then P is at most degree 3. **[2 Marks]**

By arguments identical to those above, we deduce that

$$P(x) = dL_1(x)L_2(x)L_3(x),$$

where d is a constant, and $L_3(x)$ is a non-degenerate linear polynomial that vanishes on the line intersecting z_1 and z_2. **[4 Marks]**

None of $L_i(x)$, $i = 1, 2, 3$, vanish at the midpoint $(z_1 + z_2 + z_3)/3$, but $P(x)$ does, so $d = 0$ and hence $P(x) = 0$ as required. **[4 Marks]**

3. Let K be the interval $[0,1]$, and let $\mathcal{P}$ be one-dimensional polynomials of degree 3 or less, with a dual basis $\mathcal{N}$. Let T_h be the corresponding subdivision of the interval $[a,b]$, with elements defined on each subinterval that are affine-equivalent to $(K,\mathcal{P},\mathcal{N})$.

1. Determine a dual basis $\mathcal{N}$ on K, such that the corresponding global interpolation operator $\mathcal{I}_{T_h}$ has C^1 continuity. Show that your dual basis determines $\mathcal{P}$.
   Solution: NOT SEEN
   The dual basis is point evaluation at $x=0$, $x=1$, and derivative evaluation at $x=0$ and $x=1$. **[3 Marks]**
   If P is a cubic polynomial that vanishes under the action of each dual basis element, this means that it has double roots at both $x=0$ and $x=1$. Hence, by fundamental theorem of algebra, it is identically equal to zero, and hence the dual basis determines $\mathcal{P}$. **[2 Marks]**

2. Determine the corresponding nodal basis for $\mathcal{P}$.
   Solution: NOT SEEN
   $\phi_1(x)$ has a double root at $x=1$, so it takes the value

$$\phi_1(x) = (x-1)^2(ax+b).$$

At $x=0$,

$$1 = \phi_1(0) = b.$$

We have

$$\phi_1'(x) = 2(x-1)(ax+1)+(x-1)^2a = (x-1)(2ax+2+a(x-1)) = (x-1)(3ax+2-a).$$

At $x=0$,

$$0 = \phi_1'(0) = -(2-a) \implies a = 2.$$

Hence, we get

$$\phi_1(x) = (x-1)^2(1+2x) = 2x^3 - 3x^2 + 1.$$

[4 Marks]
By symmetry, we have $\phi_2(x) = \phi_1(1-x) = x^2(1+2(1-x)) = x^2(3-2x) = 3x^2 - 2x^3$.
[1 Marks]
$\phi_3(x)$ must vanish at $x=0$ (with a double root) and $x=1$, so it takes the form

$$\phi_3(x) = cx^2(x-1).$$

Differentiating,

$$\phi_3'(x) = c\left(2x^2(x-1)+x^2\right) = c(2x^3 - x^2).$$

Evaluation of ϕ_3' at $x=1$ gives

$$1 = \phi_3'(1) = c,$$

hence

$$\phi_3(x) = x(x-1)^2 = x^3 - 2x^2 + x.$$

[4 Marks]
By symmetry, we have

$$\phi_4(x) = -\phi_3(1-x) = -(1-x)x^2 = -x^2 + x^3.$$

[1 Marks]

3. Consider the variational problem for $u \in V$,

$$\int_0^1 u''v''\,\mathrm{d}\,x = \int_0^1 fv\,\mathrm{d}\,x, \quad \forall v \in V,$$

where

$$V = \left\{ u : \int_0^1 (u'')^2\,\mathrm{d}\,x < \infty,\, u(0) = u'(0) = u(1) = u'(1) = 0 \right\}.$$

Define a corresponding finite element discretisation based on the nodal basis defined above. Assume that a unique solution u exists the variational problem, and a unique solution u_h to the corresponding finite element discretisation. Prove the Galerkin orthogonality result

$$\int_0^1 (u - u_h)''v''\,\mathrm{d}\,x = 0, \quad \forall v \in S,$$

for an appropriately defined space S.

Solution: NOT SEEN

We define the finite element space with cubic polynomials in each subdomain, and C^1 continuity between elements, and let S be the subspace satisfying the boundary conditions of the variational problem (this is possible since the finite element space allows separate specification of u and u' at each nodal point). Then, since $C^1 \subset H^2$ in 1 dimension from Sobolev's inequality, we have $S \subset V$. **[2 Marks]**

The finite element discretisation then has solution $u_h \in S$ with

$$a(u_h, v) = F(v), \quad \forall v \in S,$$

where

$$a(u, v) = \int_0^1 u''v''\,\mathrm{d}\,x, \quad F(v) = \int_0^1 vf\,\mathrm{d}\,x.$$

Since $S \subset V$, the solution the full variational problem satisfies

$$a(u, v) = F(v), \quad \forall v \in S,$$

and subtracting, we obtain

$$a(u - u_h, v) = 0, \quad \forall v \in S,$$

as required. **[3 Marks]**

4. Let K be the reference square element K with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Let $\mathcal{P}$ be the space of bilinear functions defined on K. Let $\mathcal{N}$ be the dual basis given by pointwise evaluation at each element vertex. Let $\mathcal{I}_K : C^0(K) \to \mathcal{P}$ be the interpolation operator defined by the corresponding nodal basis for $\mathcal{P}$.

We assume that it can be shown that

$$|u - \mathcal{I}_K u|_{H^1(K)} \leq c_0 \, |u|_{H^2(K)}\,, \quad \forall u \in H^2(K),$$

where $c_0 > 0$ is a constant that is independent of u, and where $|\cdot|_{H^m(K)}$ is the $H^m(K)$ seminorm on K.

1. For a mesh element K_h with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), (x_{i+1}, y_{j+1}), with $x_i = x_0 + ih$, $y_j = y_0 + jh$, show that

$$|u - \mathcal{I}_{K_h} u|_{H^1(K_h)} \leq c_0 h |u|_{H^2(K_h)}, \quad \forall u \in H^2(K_h),$$

where $\mathcal{I}_{K_h}$ is the nodal interpolation operator to bilinear polynomials defined on K_h.
Solution: SEEN SIMILAR
Squaring the interpolation result, we obtain

$$\begin{aligned}
&\int_K \left(\frac{\partial}{\partial x}(u - \mathcal{I}_K u)\right)^2 + \left(\frac{\partial}{\partial y}(u - \mathcal{I}_K u)\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y \\
\leq & c_0^2 \int_K \left(\frac{\partial^2}{\partial x^2}(u - \mathcal{I}_K u)\right)^2 + \left(\frac{\partial^2}{\partial xy}(u - \mathcal{I}_K u)\right)^2 + \left(\frac{\partial^2}{\partial y^2}(u - \mathcal{I}_K u)\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

[6 Marks] *Then we apply a change of variables*

$$x \mapsto \bar{x} = x_i + xh, \quad y \mapsto \bar{y} = y_i + yh,$$

to obtain

$$\begin{aligned}
&\int_{K_h} \left(h\frac{\partial}{\partial \bar{x}}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(h\frac{\partial}{\partial \bar{y}}(u - \mathcal{I}_{K_h} u)\right)^2 \frac{1}{h^2}\,\mathrm{d}\,\bar{x}\,\mathrm{d}\,\bar{y} \\
\leq & c_0^2 \int_K \left(\left(h^2\frac{\partial^2}{\partial \bar{x}^2}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(h^2\frac{\partial^2}{\partial \bar{x}\bar{y}}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(h^2\frac{\partial^2}{\partial \bar{y}^2}(u - \mathcal{I}_{K_h} u)\right)^2\right) \frac{1}{h^2}\,\mathrm{d}\,\bar{x}\,\mathrm{d}\,\bar{y},
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{K_h} \left(\frac{\partial}{\partial \bar{x}}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial}{\partial \bar{y}}(u - \mathcal{I}_{K_h} u)\right)^2 \mathrm{d}\,\bar{x}\,\mathrm{d}\,\bar{y} \\
\leq & c_0^2 h^2 \int_K \left(\left(\frac{\partial^2}{\partial \bar{x}^2}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial^2}{\partial \bar{x}\bar{y}}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial^2}{\partial \bar{y}^2}(u - \mathcal{I}_{K_h} u)\right)^2\right) \mathrm{d}\,\bar{x}\,\mathrm{d}\,\bar{y}.
\end{aligned}$$

Taking the square root, we obtain the result. **[8 Marks]**

2. Use this result to show that

$$|u - \mathcal{I}_h u|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)},$$

where $\mathcal{I}_h$ is the global interpolation operator to bilinear Lagrange elements defined on a mesh of a square domain Ω constructed from $h \times h$ squares.

Solution: SEEN SIMILAR

Squaring the previous result, and summing over all of the elements, we get

$$\sum_{K_h} \int_{K_h} \left(\frac{\partial}{\partial x}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial}{\partial y}(u - \mathcal{I}_{K_h} u)\right)^2 \mathrm{d}\, x \, \mathrm{d}\, y$$
$$\leq c_0^2 h^2 \sum_{K_h} \int_K \left(\left(\frac{\partial^2}{\partial x^2}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial^2}{\partial xy}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial^2}{\partial y^2}(u - \mathcal{I}_{K_h} u)\right)^2\right) \mathrm{d}\, x \, \mathrm{d}\, y,$$

and hence

$$\int_{\Omega} \left(\frac{\partial}{\partial x}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial}{\partial y}(u - \mathcal{I}_{K_h} u)\right)^2 \mathrm{d}\, x \, \mathrm{d}\, y$$
$$\leq c_0^2 h^2 \Omega \left(\left(\frac{\partial^2}{\partial x^2}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial^2}{\partial xy}(u - \mathcal{I}_{K_h} u)\right)^2 + \left(\frac{\partial^2}{\partial y^2}(u - \mathcal{I}_{K_h} u)\right)^2\right) \mathrm{d}\, x \, \mathrm{d}\, y,$$

and square root leads to the result. **[6 Marks]**