MSc EXAMINATIONS (MATHEMATICS)

XXXX 2015

M5MA47

Finite Elements: numerical analysis and implementation

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This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M5MA47

Finite Elements: numerical analysis and implementation

Date: XXXday, XX XXXXX 2015  Time: XX.00 Xm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly incomplete answers.

Calculators may not be used.
1. Provide a variational formulation for the following equation for $u$.

$$u'' - u = -f, \quad u(0) = 0, \quad u'(1) = 1.$$ 

2. A quadrature rule on a reference element $K$ has degree of precision $m$ if it produces the exact answer for all polynomials of degree $m$ or less. What is the minimum degree of precision required to exactly assemble the matrix for the bilinear form

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

on a mesh of elements that are affine equivalent to the finite element $(K, \mathcal{P}_2, \mathcal{N})$, where $\mathcal{P}_2$ are the polynomials of degree 2 or less, and $\mathcal{N}$ is the dual basis corresponding to evaluation at triangle vertices and midpoints.

3. Obtain the nodal basis function $\phi_1(x)$ for the finite element $(K = [0, 1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$N_1(f) = f'(0),$$

$$N_2(f) = f'(1),$$

$$N_3(f) = \int_0^1 f \, dx.$$

4. What is the global continuity of finite element spaces constructed from the finite element described in part (3) of this question? Explain your answer.
1. Consider the finite element \((K, P, N)\) where

- \(K\) is a non-degenerate triangle.
- \(P\) is the space of polynomials of degree 2 or less.
- \(N = (N_1, N_2, N_3, N_4, N_5, N_6)\) with
  
  \[
  N_i(v) = v(z_i), \quad i = 1, 2, 3, \\
  N_4(v) = v\left(\frac{z_1 + z_2}{2}\right), \\
  N_5(v) = v\left(\frac{z_1 + z_3}{2}\right), \\
  N_6(v) = v\left(\frac{z_2 + z_3}{2}\right),
  \]

  where \(z_1, z_2\) and \(z_3\) are the vertices of \(K\).

Show that \(N\) determines \(P\).

2. Now consider the finite element \((K, \hat{P}, \hat{N})\) where

- \(K\) is a non-degenerate triangle (with boundary \(\partial K\)).
- \(\hat{P}\) is the space spanned by polynomials of degree 2 or less, plus the cubic “bubble” function \(B(x)\) satisfying \(B(x) = 0\) for all \(x \in \partial K\), and \(\int_K B(x) \, dx = 1\).
- \(\hat{N} = (N_1, N_2, N_3, N_4, N_5, N_6, N_7)\) with \(N_i\) as above for \(i = 1, \ldots, 6\), and
  
  \[
  N_7(v) = v\left(\frac{z_1 + z_2 + z_3}{3}\right).
  \]

Show that \(\hat{N}\) determines \(\hat{P}\).
3. Let $K$ be the interval $[0, 1]$, and let $\mathcal{P}$ be one-dimensional polynomials of degree 3 or less, with a dual basis $\mathcal{N}$. Let $T_h$ be the corresponding subdivision of the interval $[a, b]$, with elements defined on each subinterval that are affine-equivalent to $(K, \mathcal{P}, \mathcal{N})$.

1. Determine a dual basis $\mathcal{N}$ on $K$, such that the corresponding global interpolation operator $I_{T_h}$ has $C^1$ continuity. Show that your dual basis determines $\mathcal{P}$.

2. Determine the corresponding nodal basis for $\mathcal{P}$.

3. Consider the variational problem for $u \in V$,
\[
\int_0^1 u'' v'' \, dx = \int_0^1 f v \, dx, \quad \forall v \in V,
\]
where
\[
V = \left\{ u : \int_0^1 (u'')^2 \, dx < \infty, \, u(0) = u'(0) = u(1) = u'(1) = 0 \right\}.
\]
Define a corresponding finite element discretisation based on the nodal basis defined above. Assume that a unique solution $u$ exists the variational problem, and a unique solution $u_h$ to the corresponding finite element discretisation. Prove the Galerkin orthogonality result
\[
\int_0^1 (u - u_h)'' v'' \, dx = 0, \quad \forall v \in S,
\]
for an appropriately defined space $S$. 

4. Let $K$ be the reference square element $K$ with vertices $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$. Let $P$ be the space of bilinear functions defined on $K$. Let $\mathcal{N}$ be the dual basis given by pointwise evaluation at each element vertex. Let $I_K : C^0(K) \rightarrow P$ be the interpolation operator defined by the corresponding nodal basis for $P$.

We assume that it can be shown that

$$|u - I_K u|_{H^1(K)} \leq c_0 |u|_{H^2(K)}, \quad \forall u \in H^2(K),$$

where $c_0 > 0$ is a constant that is independent of $u$, and where $| \cdot |_{H^m(K)}$ is the $H^m(K)$ seminorm on $K$.

1. For a mesh element $K_h$ with vertices $(x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), (x_{i+1}, y_{j+1})$, with $x_i = x_0 + ih$, $y_j = y_0 + jh$, show that

$$|u - I_{K_h} u|_{H^1(K_h)} \leq c_0 h |u|_{H^2(K_h)}, \quad \forall u \in H^2(K_h),$$

where $I_{K_h}$ is the nodal interpolation operator to bilinear polynomials defined on $K_h$.

2. Use this result to show that

$$|u - I_h u|_{H^1(\Omega)} \leq C h |u|_{H^2(\Omega)},$$

where $I_h$ is the global interpolation operator to bilinear Lagrange elements defined on a mesh of a square domain $\Omega$ constructed from $h \times h$ squares.