Imperial College London

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MSc EXAMINATIONS (MATHEMATICS)

XXXX 2015

M5MA47

Finite Elements: numerical analysis and implementation

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MSc EXAMINATIONS (MATHEMATICS)

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This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M5MA47

Finite Elements: numerical analysis and implementation

Date: XXXday, XX XXXXX 2015 Time: XX.00 Xm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly incomplete answers.

Calculators may not be used.

1. 1. Provide a variational formulation for the following equation for u.

$$u'' - u = -f, \quad u(0) = 0,\, u'(1) = 1.$$

2. A quadrature rule on a reference element K has degree of precision m if it produces the exact answer for all polynomials of degree m or less. What is the minimum degree of precision required to exactly assemble the matrix for the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx,$$

on a mesh of elements that are affine equivalent to the finite element $(K, \mathcal{P}_2, \mathcal{N})$, where $\mathcal{P}_2$ are the polynomials of degree 2 or less, and $\mathcal{N}$ is the dual basis corresponding to evaluation at triangle vertices and midpoints.

3. Obtain the nodal basis function $\phi_1(x)$ for the finite element $(K = [0, 1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$N_1(f) = f'(0),$$
$$N_2(f) = f'(1),$$
$$N_3(f) = \int_0^1 f dx.$$

4. What is the global continuity of finite element spaces constructed from the finite element described in part (3) of this question? Explain your answer.

2. 1. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

   – K is a non-degenerate triangle.
   – $\mathcal{P}$ is the space of polynomials of degree 2 or less.
   – $\mathcal{N} = (N_1, N_2, N_3, N_4, N_5, N_6)$ with

$$N_i(v) = v(z_i),\ i = 1, 2, 3,$$
$$N_4(v) = v\left(\frac{z_1 + z_2}{2}\right),$$
$$N_5(v) = v\left(\frac{z_1 + z_3}{2}\right),$$
$$N_6(v) = v\left(\frac{z_2 + z_3}{2}\right),$$

   where z_1, z_2 and z_3 are the vertices of K.

   Show that $\mathcal{N}$ determines $\mathcal{P}$.

2. Now consider the finite element $(K, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ where

   – K is a non-degenerate triangle (with boundary ∂K).
   – $\hat{\mathcal{P}}$ is the space spanned by polynomials of degree 2 or less, plus the cubic "bubble" function $B(x)$ satisfying $B(x) = 0$ for all $x \in \partial K$, and $\int_K B(x)\,\mathrm{d}\,x = 1$.
   – $\hat{\mathcal{N}} = (N_1, N_2, N_3, N_4, N_5, N_6, N_7)$ with N_i as above for $i = 1, \ldots, 6$, and

$$N_7(v) = v\left(\frac{z_1 + z_2 + z_3}{3}\right).$$

   Show that $\hat{N}$ determines $\hat{P}$.

3. Let K be the interval $[0,1]$, and let $\mathcal{P}$ be one-dimensional polynomials of degree 3 or less, with a dual basis $\mathcal{N}$. Let T_h be the corresponding subdivision of the interval $[a,b]$, with elements defined on each subinterval that are affine-equivalent to $(K,\mathcal{P},\mathcal{N})$.

1. Determine a dual basis $\mathcal{N}$ on K, such that the corresponding global interpolation operator $\mathcal{I}_{T_h}$ has C^1 continuity. Show that your dual basis determines $\mathcal{P}$.
2. Determine the corresponding nodal basis for $\mathcal{P}$.
3. Consider the variational problem for $u \in V$,

$$\int_0^1 u''v''\,\mathrm{d}\,x = \int_0^1 fv\,\mathrm{d}\,x, \quad \forall v \in V,$$

where

$$V = \left\{u : \int_0^1 (u'')^2\,\mathrm{d}\,x < \infty,\, u(0) = u'(0) = u(1) = u'(1) = 0\right\}.$$

Define a corresponding finite element discretisation based on the nodal basis defined above. Assume that a unique solution u exists the variational problem, and a unique solution u_h to the corresponding finite element discretisation. Prove the Galerkin orthogonality result

$$\int_0^1 (u - u_h)''v''\,\mathrm{d}\,x = 0, \quad \forall v \in S,$$

for an appropriately defined space S.

4. Let K be the reference square element K with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Let $\mathcal{P}$ be the space of bilinear functions defined on K. Let $\mathcal{N}$ be the dual basis given by pointwise evaluation at each element vertex. Let $\mathcal{I}_K : C^0(K) \to \mathcal{P}$ be the interpolation operator defined by the corresponding nodal basis for $\mathcal{P}$.

We assume that it can be shown that

$$|u - \mathcal{I}_K u|_{H^1(K)} \leq c_0 \, |u|_{H^2(K)} \, , \quad \forall u \in H^2(K),$$

where $c_0 > 0$ is a constant that is independent of u, and where $|\cdot|_{H^m(K)}$ is the $H^m(K)$ seminorm on K.

1. For a mesh element K_h with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), (x_{i+1}, y_{j+1}), with $x_i = x_0 + ih$, $y_j = y_0 + jh$, show that

$$|u - \mathcal{I}_{K_h} u|_{H^1(K_h)} \leq c_0 h |u|_{H^2(K_h)}, \quad \forall u \in H^2(K_h),$$

where $\mathcal{I}_{K_h}$ is the nodal interpolation operator to bilinear polynomials defined on K_h.

2. Use this result to show that

$$|u - \mathcal{I}_h u|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)},$$

where $\mathcal{I}_h$ is the global interpolation operator to bilinear Lagrange elements defined on a mesh of a square domain Ω constructed from $h \times h$ squares.